

OPTIMIZATION WITH MIXED-INTEGER, COMPLEMENTARITY,
AND BILEVEL CONSTRAINTS WITH APPLICATIONS TO ENERGY
AND FOOD MARKETS

by

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Abstract

In this dissertation, we discuss three classes of nonconvex optimization problems, namely, mixed-integer programming, nonlinear complementarity problems, and mixed-integer bilevel programming.

For mixed-integer programming, we identify a class of cutting planes, namely the class of cutting planes derived from lattice-free cross-polytopes, which are proven to provide good approximations to the problem while being efficient to compute. We show that the closure of these cuts gives an approximation that depends only on the ambient dimension and that the cuts can be computed efficiently by explicitly providing an algorithm to compute the cut coefficients in $O(n2^n)$ time, as opposed to solving a nearest lattice-vector problem, which could be much harder.

For complementarity problems, we develop a first-order approximation algorithm to efficiently approximate the covariance of the decision in a stochastic complementarity problem. The method can be used to approximate the covariance for large-scale problems by solving a system of linear equations. We also provide bounds to the error incurred in this technique. We then use the technique to analyze policies related to the North American natural gas market.

Further, we use this branch of nonconvex problems in the Ethiopian food market to analyze the regional effects of exogenous shocks on the market. We develop a detailed model of the food production, transportation, trade, storage, and consumption in Ethiopia, and test it against exogenous shocks. These shocks are motivated by the prediction that teff, a food grain whose export is banned now, could become a super grain. We present the regional effects of different government policies in response to this shock.

For mixed-integer bilevel programming, we develop algorithms that run in polynomial time, provided a subset of the input parameters are fixed. Besides the Σ_2^P -hardness of the general version of the problem, we show polynomial solvability and NP -completeness of certain restricted versions of this problem.

Finally, we completely characterize the feasible regions represented by each of these different types of nonconvex optimization problems. We show that the representability of linear complementarity problems, continuous bilevel programs, and polyhedral reverse-convex programs are the same, and they coincide with that of mixed-integer programs if the feasible region is bounded. We also show that the feasible region of any mixed-integer bilevel program is a union of the feasible regions of finitely many mixed-integer programs up to projections and closures.

Readers: Sauleh Siddiqui, Amitabh Basu, Michael Shields

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Chapter 1

Introduction and motivation

The problem of finding a point where a function takes the largest (or smallest) possible value, given some conditions that such a point should satisfy is an *optimization problem*. The function which has to be maximized or minimized is called the *objective* and the restrictions on the set of points one is allowed to choose from are called the *constraints*. The problem of optimizing a *convex* function over a *convex* set is a well-understood problem and in general, we have reasonably efficient algorithms to solve the problem to global optimality. Though convex optimization problems model a reasonable subset of problems of interest, a vast majority of problems beg for more sophisticated modeling approaches. Making no assumptions about convexity enables us to model a significantly large class of problems, but such problems are significantly harder, sometimes even impossible, i.e., Turing undecidable [88, for example] to solve to global optimality. This dissertation introduces methods that strike a balance between the ability to model larger classes of problems compared to convex optimization problems while retaining the ability to solve a large class of problems efficiently.

In particular, this dissertation deals with three different classes of nonconvex problems, namely, (i) mixed-integer linear programs (ii) complementarity problems and (iii) mixed-integer bilevel

programs.

Integer nonconvexity, which is the restriction that a subset of the decision variables must be integers, is one of the most extensively studied subfields of nonconvexities. Conforti, Cornuéjols, and Zambelli [43], Nemhauser and Wolsey [118], Schrijver [126] are some references where this subbranch is covered in great detail.

Nonconvexity arising due to complementarity constraints, which we refer to as complementarity nonconvexity, requires two vectors of the same size to be component-wise non-negative, and their inner product to evaluate to zero. In other words, the said vectors should have non-negative components, and for each component, the corresponding element in at least one of the two vectors should be exactly zero. The case where these two vectors are related by a linear function is studied extensively in Cottle, Pang, and Stone [47] and the case where they could be related by a nonlinear function is studied in Facchinei and Pang [66, 67].

Mixed-integer bilevel programs arise naturally from a problem in economics known as the Stackelberg game [37]. This requires that a subset of the solution variables of one optimization should also be a solution of *another* optimization problem, which is parameterized in the remaining variables. This induces nonconvexity in the optimization problem and we refer to such nonconvexity as mixed-integer bilevel nonconvexity.

In the rest of this chapter, we formally define convexity and motivate the need to methodically include nonconvex constraints. Then we motivate the applicability of complementarity constraints in two areas, namely the North American natural gas market and Ethiopian food market. [Chapter 2](#) has definitions and previously-known results that are preliminary to the results in [Chapter 3](#) to [6](#). Following that, [Chapter 3](#) deals with the nonconvexities in the form of integer constraints, and develops new techniques that fall under the so-called cutting plane approach to the problem.

Theoretical, as well as practical evaluation of the new tools, is considered in [Chapter 3](#). [Chapter 4](#) talks about the two different real-world problems arising from the North American natural gas market and Ethiopian food market, that we modeled using complementarity constraints. One of the applications also describes a novel algorithm to approximate the covariance of the solution of a complementarity problem, under uncertainty of problem parameters. [Chapter 5](#) describes a new algorithm to solve a relaxation of the mixed-integer bilevel programs under polynomial time, provided a subset of the problem parameters are fixed. Finally, [Chapter 6](#) comprehensively characterizes and quantifies the set of problems that can be modeled using the said types of nonconvexities and [Chapter 7](#) concludes the dissertation with some ideas and directions for future work. The dissertation also has appendices to provide supplementary details. [Appendix A](#) has a key to the set of standard notations used across multiple chapters in the dissertation. [Appendix B](#) provides the comprehensive set of equations defining the North American natural gas model. [Appendix C](#) provides details about a python package we developed and shared to handle multi-dimensional sparse arrays. [Appendix D](#) provides the comprehensive set of equations defining the DECO2 model we use to analyze the Ethiopian food market and [Appendix E](#) lists the set of papers published, submitted and in preparation as a part of the doctoral work.

Conventions: In this dissertation, we refer to *solving* an optimization problem as finding the global optimum of the problem. Local minima are not considered unless explicitly stated otherwise.

1.1 The assumptions of convexity

In order to make formal statements about including non-convex constraints, it is necessary to mathematically state what the assumption of convexity means and the type of problems that can

be solved as a convex optimization problem.

Definition 1 (Convex set). *A set $C \subseteq \mathbb{R}^n$ is a convex set if $\forall x, y \in C$ and $\forall \eta \in [0, 1]$, $\eta x + (1 - \eta)y \in C$.*

A set that is not convex is called a *nonconvex* set. Fig. 1.1a and 1.1b show an example of a convex and a nonconvex set respectively.

Definition 2 (Convex function). *A function $f : C \mapsto \mathbb{R}$ is a convex function, if the domain C is a convex set and $\forall x, y \in C$, and $\forall \eta \in [0, 1]$,*

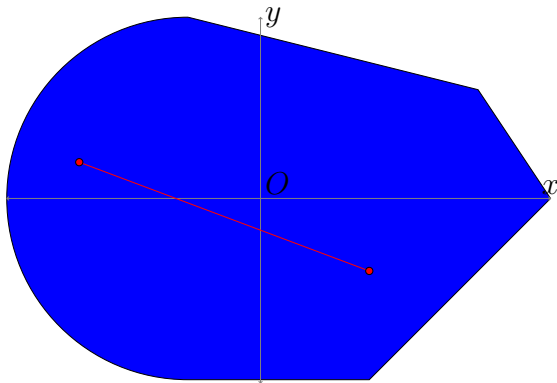
$$f(\eta x + (1 - \eta)y) \leq \eta f(x) + (1 - \eta)f(y)$$

A function that is not convex is called a *nonconvex* function. Fig. 1.1c and 1.1d show an example of a convex and a nonconvex function respectively.

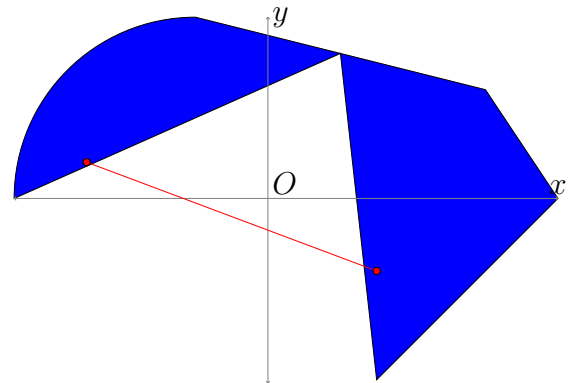
Definition 3 (Optimization problem). *Given $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $S \subseteq \mathbb{R}^n$, the optimization problem is to solve $\min_{x \in S} f(x)$. We call the set S as the feasible set and the function f as the objective function. If $S = \mathbb{R}^n$, then we call the problem an unconstrained optimization problem.*

Note that a minimum as necessitated in the previous definition need not necessarily exist. Even under some natural cases, for example, the mixed-integer bilevel linear program, it is possible to construct examples where S is not necessarily closed. In such cases, we typically either report that the minimum does not exist or work with the infimum with the caveat that such an infimum cannot be achieved by any feasible point. But an explicit mentioning of the convention is always made in this dissertation.

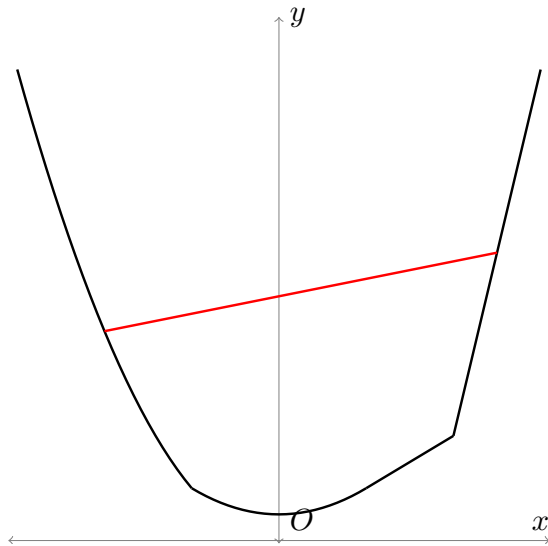
We now define certain standard terms used in convex analysis. Let $S \subseteq \mathbb{R}^n$. Then the *convex hull* of S is $\text{conv}(S) := \left\{ x = \sum_{i=1}^k \lambda_i v^i : 0 \leq \lambda_i \leq 1, \sum_{i=1}^k \lambda_i = 1, v^i \in S \right\}$. In other



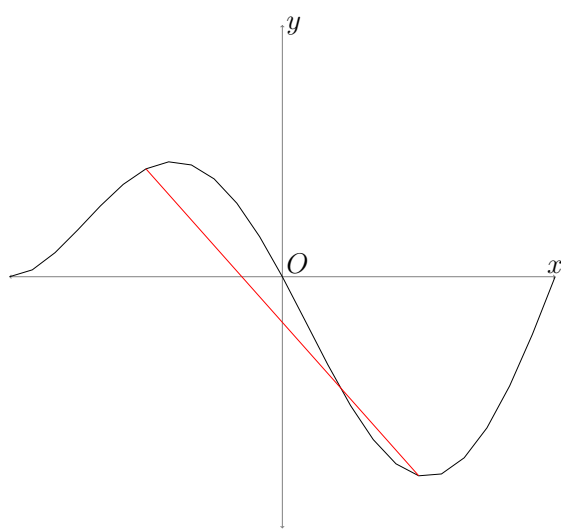
(a) The blue shaded region is a convex set in \mathbb{R}^2 . Note that any line connecting two points in this set is contained in the set.



(b) The blue shaded region is a nonconvex set in \mathbb{R}^2 . Note that the red line connecting two points in the set is not contained in the set.



(c) A convex function. Note that the function always stays below any secant like the one shown in red.



(d) A nonconvex function $\mathbb{R} \mapsto \mathbb{R}$. Note that the function doesn't stay necessarily below the secant shown in red.

Figure 1.1: Convex and nonconvex functions and sets

words it is the inclusion-wise smallest convex set containing S . A set C is called a *cone* if $x \in C, y \in C \implies \lambda x + \mu y \in C, \forall \lambda, \mu \in \mathbb{R}_+$. Note that this definition implies that all cones are convex sets. The *conic hull* of S is $\text{cone}(S) := \left\{x = \sum_{i=1}^k \lambda_i v^i : 0 \leq \lambda_i, v^i \in S\right\}$. In other words, it is the inclusion-wise smallest cone containing S . The *affine hull* of S is define as $\text{aff}(S) := \left\{x = \sum_{i=1}^k \lambda_i v^i : \sum_{i=1}^k \lambda_i = 1, v^i \in S\right\}$. Suppose S is a convex set, the *relative interior* of S is defined as $\text{relint}(S) := \{x \in S : \exists \varepsilon > 0 \text{ such that } B(x, \varepsilon) \cap \text{aff}(S) \subseteq S\}$. Let $A, B \subseteq \mathbb{R}^n$. Then their *Minkowski sum* $A + B := \{x + y : x \in A, y \in B\}$. Similarly we define $\alpha A := \{\alpha x : x \in A\}$ for any $\alpha \in \mathbb{R}$. We define $A - B := A + (-1)B$.

Definition 4 (Feasibility problem). *Given $S \subseteq \mathbb{R}^n$ we define the feasibility problem of S as follows: either find a vector $v \in S$ or report that $S = \emptyset$. If $S \neq \emptyset$, the solution $v \in S$ is an example of a certificate of feasibility.*

Remark 5. Note that the feasibility problem and the optimization problem are closely related. Given an optimization problem, one can always ask a series of “a few” feasibility questions that enable to obtain the optimum with arbitrary precision [85].

Definition 6 (Convex optimization problem). *A convex optimization problem is $\min_{x \in C} f(x)$ where C is a convex set and f is a convex function.*

Convexity is an elegant assumption in optimization problems that greatly enhances the solvability of the problem. The definition of a convex function motivates a U-shaped function, whose every local minimum is guaranteed to be a global minimum and the set of global minima is also a convex set. This structure enables us to design efficient algorithms to solve convex optimization problems. On the other hand, an absence of such structure can make a problem quite hard to solve. Here we reformulate a problem from number theory which was unsolved for centuries into a “small” nonconvex optimization problem.

Example 7. ¹ Fermat's last theorem states that no three positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer $n \geq 3$. The conjecture was made in 1637 and the proof remained elusive for over three centuries despite continued efforts. It was finally proved in 1994, about 358 years after the conjecture was made. Now, consider the following continuous nonconvex optimization problem.

$$\min_{a, b, c, n \in \mathbb{R}} : (c^n - a^n - b^n)^2$$

subject to

$$a, b, c \geq 1$$

$$n \geq 3$$

$$\sin^2(a\pi) + \sin^2(b\pi) + \sin^2(c\pi) + \sin^2(n\pi) \leq 0$$

This problem is a “small” nonconvex optimization problem with 4 variables and 5 constraints. The objective function, the constraints are all smooth. The objective is 0 if and only if $c^n = a^n + b^n$, while the last constraint ensures that each of the four variables are integers, as integer multiples of π are the only points where the sin function vanishes. Now, if one can prove that the optimal objective value of the problem is strictly greater than 0, then they have proved Fermat's last theorem. Conversely, if one proves that the optimal objective value of the problem is equal to zero, they have disproved Fermat's last theorem.

This example motivates how hard even a reasonably small (counting the number of variables and constraints) nonconvex optimization problem can be. Thus assumptions of convexity can be quite useful.

¹Thanks to the lecture notes of Dr. Amir Ali Ahmadi, Princeton University for this example.

While we state this, it is not true that every nonconvex optimization problem is very difficult to solve or that every convex optimization problem is easy to solve. For example, the S-lemma [121, 141] enables one to write a quadratic program with a single quadratic constraint (i.e., problems where $f(x) = x^T Q_0 x + c_0^T x$ and $S = \{x : x^T Q_1 x + c_1^T x + v_1 \leq 0\}$ in the notation of Definition 3) equivalently as a semidefinite program. No assumptions of convexity are required on the objective or the constraint. This nonconvex optimization problem can be solved with any desired accuracy in polynomial time. Thus it serves as an example of a computationally tractable nonconvex optimization problem. On the flip-side, checking non-emptiness of the *convex* copositive cone (A matrix $M \in \mathbb{R}^{n \times n}$ is copositive if $x^T M x \geq 0$ for every non-negative $x \in \mathbb{R}^n$) under finitely many affine constraints is NP-complete [35, 117]. So unless $P = NP$, we have no hope of solving this *convex* optimization problem in polynomial time.

Despite the observation that there are convex optimization problems which are computationally intractable and nonconvex optimization problems that are efficiently solvable, there is a non-trivial overlap between the set of computationally tractable optimization problems and the set of convex optimization problems. Thus in the literature and modern research, convex optimization is considered synonymous to the computationally tractable class of optimization problems.

However, the assumptions on convexity limit our modeling power. We call the representability (a more formal term for modeling power) of a given family of sets as the larger family of sets which can be represented by the first family potentially using additional variables and projections. We define representability more formally below.

Definition 8 (Representability). *Let \mathcal{T} be a given family of sets. We say that a set S is represented by \mathcal{T} or \mathcal{T} -representable, if S is a linear transformation of a set in \mathcal{T} . We denote the family of all \mathcal{T} -representable sets as \mathcal{T}_R .*

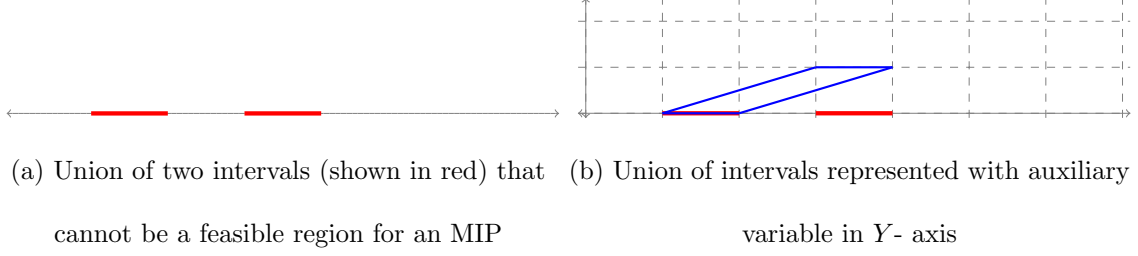


Figure 1.2: Representability of mixed-integer program (MIP) - A union of two intervals can be represented using auxiliary variables in an MIP

[Definition 8](#) is motivated to answer questions about the type of feasible regions we can model with a family. Below, we give an example of a set that is not a feasible region of any mixed-integer program but can still be represented using a mixed-integer program using additional variables.

Example 9. Suppose we are interested in optimizing a function f over the union of two intervals, $[1, 2] \cup [3, 4]$. The feasible region is shown in red in [Figure 1.2a](#). We cannot directly write this as a feasible region of an integer program defined in [Definition 10](#). However, suppose we use auxiliary variables, consider the following set of constraints:

$$\begin{aligned} y &\geq 0 & y &\leq 1 \\ x - 2y &\geq 1 & x - 2y &\leq 2 \end{aligned}$$

The polyhedron defined by these equations correspond to the region bounded by blue lines in [Figure 1.2b](#). We use an auxiliary variable y . Now, if we constrain that y should be an integer, i.e., $y \in \mathbb{Z}$, then the projection of this set onto the space of x variables, gives the required region $[1, 2] \cup [3, 4]$. Here we observe that, while a mixed-integer linear program (MIP) cannot directly model the union of two intervals, using auxiliary variables which can be later discarded, helped us to model the set of interest. Here, we just showed that $[1, 2] \cup [3, 4]$ is MIP-representable. More generally, characterizing representability helps us understand the family of sets which can be

obtained using auxiliary variables.

It is not too difficult to see that a projection (or a linear transform) of a convex set is always a convex set [123]. Hence any linear transform of a feasible set of a convex optimization problem is a convex set. Thus it becomes necessary to include nonconvex constraints to optimize over nonconvex sets. And indeed, as seen in the above example, allowing nonconvexity, especially with projections, enabled us to model more complicated nonconvex feasible sets.

However as said before, the nonconvexity can increase the difficulty of solving the problem. We want to simultaneously increase the representability of our class of problems diligently without compromising on the complexity of solving it. Chapter 3 is concerned with improving the efficiency of algorithms for an MIP. Chapter 5 guarantees efficient approximate algorithms for general bilevel programs which have polynomial time guarantees when some of the parameters are fixed. Chapter 6 completely characterizes the representability of both linear complementarity problems (LCP) and mixed-integer bilevel programs (MIBLP) extending the previously known characterization for MIPs due to Jeroslow and Lowe [92].

1.2 Applications of nonconvex optimization

In the previous section, we showed an example of a set that can only be modeled using nonconvex constraints. Using these nonconvex constraints help us model practical problems, in addition to the theoretical curiosities that they resolve. In this dissertation, we show the application of complementarity constraints to model economic equilibrium.

In a market, economic equilibrium is the point at which all players in the market have no incentive to change their economic decisions (See Definition 16 for a more formal definition). Under mild assumptions, it can be shown that the market players eventually reach an equilibrium.

Understanding the market equilibrium helps in designing policies and interventions in the market. Complementarity constraints help in identifying the equilibrium. By computing the equilibrium under different scenarios, one quantitatively analyze changes in welfare under these scenarios.

In this dissertation, we show the applicability of complementarity problems to economic equilibrium problems arising in the North American natural gas market and the Ethiopian food market. We discuss the motivation for our study in the forthcoming subsections.

1.2.1 North American Natural Gas Model (NANGAM)

New technology in the field of natural gas exploration, known as hydraulic fracturing technology and horizontal drilling, led to a large increase in United States natural gas production. This *shale boom* and the announcements by the United States Environment Protection Agency (US-EPA) to impose new power plant regulations to control greenhouse gas emissions, led to a lot of academic debates and research about the role of natural gas in North America over the first half of the 21st century [69]. Our understanding of the crucial role of Mexico and Canada in the natural gas market is particularly limited. Feijoo et al. [69] observe that the demand for natural gas grew by over 60% in Mexico between 2005 and 2015. They further note that increased consumption from the electricity sector is the primary cause of this growth in demand. Feijoo et al. [69] developed a model using complementarity constraints to identify the market equilibrium under different energy policies by the Mexican government. The model was constructed with deterministic parameters for futuristic values for the demand curve, the supply curve, and costs of transportation and infrastructure expansion.

In the work described in this dissertation, we solve the complementarity problem under uncertainty in the futuristic parameters and efficiently obtain an approximation of the second-

order statistical information of the solution. This gives us information about the variability of the equilibrium obtained and the correlation between variables in the solution.

Additionally, we also introduce a sensitivity metric which quantifies the change in uncertainty in the output due to a perturbation in the variance of uncertain input parameters. This helps us to directly compare input parameters by the amount of uncertainty they propagate to the solution.

1.2.2 Ethiopian Food market

Ethiopia is an East African country whose economy is predominantly based on agriculture. The food-price crisis in 2008 led to the country banning teff exports, with an intention to insulate domestic food prices from global food price. Most of the analyses so far, retrospectively discuss the impacts of the governmental policy on the nation as a whole in a macroeconomic level. Instead, a useful tool for the policy-maker would be something that allows her to compare the effects of global as well as policy shocks on a regionalized basis. The action of the market agents under different policy scenarios and different global economic shocks can be modeled with adequate regionalized details using complementarity problems. We discuss the details of such modeling and the policy insights obtained hence in [Chapter 4](#).

Chapter 2

Preliminaries

In this chapter, we state a few preliminary definitions, theorems and other previously known results which frequent throughout the dissertation.

We now formally define the types of nonconvex constraints we consider in this dissertation. We define them in the feasibility version of the problem and say that the optimization version of any of these is just minimizing a linear function over the given set whose feasibility is checked.

Definition 10 (Mixed-integer program). *Given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $\mathcal{I} \subseteq [n]$, the mixed-integer program (MIP) is to find an element in the set*

$$\{x \in \mathbb{R}^n : Ax \leq b, x_i \in \mathbb{Z} \text{ if } i \in \mathcal{I}\}$$

or to show that the set is empty.

We say that the mixed-integer program is in *standard form*, if the set in [Definition 10](#) is presented in the form $\{x \in \mathbb{R}^n : Ax = b, x \geq 0, x_i \in \mathbb{Z} \text{ if } i \in \mathcal{I}\}$. Further if $\mathcal{I} = [n]$, we call the problem as a *pure integer program* and if $\mathcal{I} = \emptyset$, the problem is a linear programming problem. We refer the readers to Conforti et al. [\[43\]](#), Nemhauser and Wolsey [\[118\]](#), Schrijver [\[126\]](#) for a detailed treatment

of mixed-integer programming. Finally, the family of these sets whose non-emptiness is being checked in a mixed-integer program would be referred to as *mixed-integer sets*.

Definition 11 (Linear complementarity problem). *Given $M \in \mathbb{Q}^{n \times n}$ and $q \in \mathbb{Q}^n$, the linear complementarity problem (LCP) is to find an element in the set*

$$\{x \in \mathbb{R}^n : x \geq 0, Mx + q \geq 0, x^T(Mx + q) = 0\}$$

or to show that the set is empty. For convenience, we notate the above condition as $0 \leq x \perp Mx + q \geq 0$ where \perp implies that the inner product of the symbols on either of its sides should be zero.

The notion of a complementarity problem can be generalized to a *nonlinear complementarity problem*, by checking, for a given $F : \mathbb{R}^n \mapsto \mathbb{R}^n$, if the set $\{x \in \mathbb{R}^n : x \geq 0, F(x) \geq 0, x^T F(x) = 0\}$ is nonempty. We refer the readers to Cottle et al. [47] for linear complementarity problems and Facchinei and Pang [66, 67] for nonlinear complementarity problems (NCP).

Definition 12 (Mixed-integer bilevel program). *Given $A \in \mathbb{Q}^{m_\ell \times n_\ell}$, $B \in \mathbb{Q}^{m_\ell \times n_f}$, $b \in \mathbb{Q}^{m_\ell}$, $f \in \mathbb{Q}^{n_f}$, $D \in \mathbb{Q}^{m_f \times n_f}$, $C \in \mathbb{Q}^{m_f \times n_\ell}$, $g \in \mathbb{Q}^{m_f}$, $\mathcal{I}_L \subseteq [n_\ell]$, $\mathcal{I}_F \subseteq [n_f]$, define*

$$S = S^1 \cap S^2 \cap S^3 \tag{2.1a}$$

$$S^1 = \{(x, y) \in \mathbb{R}^{n_\ell + n_f} : Ax + By \leq b\} \tag{2.1b}$$

$$S^2 = \left\{ (x, y) \in \mathbb{R}^{n_\ell + n_f} : y \in \arg \max_y \{f^T y : Dy \leq g - Cx, y_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_F\} \right\} \tag{2.1c}$$

$$S^3 = \{(x, y) \in \mathbb{R}^{n_\ell + n_f} : x_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_L\} \tag{2.1d}$$

The mixed-integer bilevel program (MIBLP) is to find an element in the set S or to show that the set S is empty. We call the variables denoted by x as upper-level variables or leader's variables and we call the variables denoted by y as lower-level variables or follower's variables. We call the

constraint defined by [eq. \(2.1c\)](#) as the bilevel constraint. If $\mathcal{I}_L = \mathcal{I}_F = \emptyset$, i.e., if all variables are continuous, then we call the problem a continuous bilevel program (CBL). Further if $\mathcal{I}_F = \emptyset$, i.e., all follower's variables are continuous, then we call the problem a bilevel program with upper-level integer (BLP-UI).

We refer the reader to Bard [\[16\]](#), Dempe [\[52\]](#), Deng [\[54\]](#), Fischetti et al. [\[72, 73\]](#), Köppe et al. [\[98\]](#), Lodi et al. [\[105\]](#), Tahernejad et al. [\[134\]](#), Wang and Xu [\[138\]](#), Yue et al. [\[143\]](#) for more information on MIBLPs.

We now define some standard terms in convex analysis which are used throughout the dissertation.

Definition 13. *In the below definitions $C \subseteq \mathbb{R}^n$ is a convex set.*

- (i) Halfspace: A set of the form $\{x \in \mathbb{R}^n : \langle a, x \rangle \leq b\}$. If $a \in \mathbb{Q}^n$ and $b \in \mathbb{Q}$ we call it a rational halfspace.
- (ii) Polyhedron: An intersection of finitely many halfspaces. If each of those halfspaces are rational, we call it a rational polyhedron.
- (iii) Polytope: A polyhedron that is bounded.
- (iv) Dimension of an affine set $A \subseteq \mathbb{R}^n$: Let $x \in A$. Then $\dim(A)$ is the dimension of the linear subspace $A - \{x\}$.
- (v) Dimension of C : Dimension of $\text{aff}(C)$.
- (vi) Face of C : $F \subseteq C$ is a face of C if $x \in F$, $\forall x_1, x_2 \in C$ such that $x = \frac{x_1 + x_2}{2}$, then $x_1 \in F$, $x_2 \in F$.
- (vii) Facet of C : Suppose $\dim(C) = d$. Then a face F of C is called a facet, if $\dim(F) = d - 1$.

(viii) Vertex of C : $x \in C$ is a vertex of C if $x_1, x_2 \in C$ such that $x = \frac{x_1+x_2}{2}$, implies that

$$x_1 = x_2 = x.$$

(ix) Recession direction of C : $r \in \mathbb{R}^n$ such that $x \in C \implies x + \lambda r \in C$ for any $\lambda \geq 0$.

(x) Recession cone of C : $\{r \in \mathbb{R}^n : r \text{ is a recession direction of } C\}$. This is denoted by $\text{rec}(C)$.

(xi) Projection of a point $\text{Proj}_C(x)$: Let $x \in \mathbb{R}^n$. Then $\text{Proj}_C(\cdot)$ is the unique solution to the following convex optimization problem. $y = \arg \min_y \{\|y - x\| : y \in C\}$. In other words, $\text{Proj}_C(x)$ is the point in C that is closest to y .

(xii) Projection of a set $\text{Proj}_C(S)$: Let $S \subseteq \mathbb{R}^n$. Then $\text{Proj}_C(S) = \{y \in C : \exists x \in S \text{ such that } y = \text{Proj}_C(x)\}$.

Note that in (xi) and (xii), C is assumed to be closed, in addition to being convex.

2.1 Complexity of problem classes

We now state the following theorem about the computational complexity of each of these families of problems. Informally, we describe some terminology related to computational complexity and refer the reader to Arora and Barak [9], Jeroslow [94] for a more comprehensive and formal treatment of this subject. This study is motivated by the notion that problems that can be solved using polynomially many elementary operations (polynomial in the size of the problem description) are “easy” to solve. P is defined as the class of decision problems which can be solved using polynomially many elementary operations. NP is defined as the class of decision problems whose “yes” instances have a so-called certificate that can be verified using polynomially many elementary operations. If X is a complexity class, we say a problem Q is X -complete, if Q is in X and any problem in X can be reduced to Q using polynomially many elementary operations. The class NP is equivalently

referred to as Σ_1^P . We now define Σ_{i+1}^P as the class, whose “yes” instances have a certificate that can be verified using polynomially many calls to a problem that is Σ_i^P -complete. While it is easy to see that $P \subseteq NP \subseteq \Sigma_2^P \subseteq \dots \subseteq \Sigma_k^P \subseteq \dots$, we do not have a proof if any of these inclusions is strict. However, it is strongly conjectured that *all* these inclusions are strict, thus motivating a so-called *polynomial heirarchy* of complexity classes.

Theorem 14 (Hardness of problems). *(i) [97] Linear programming is in P .*

(ii) [45] Mixed-integer linear programming is NP-complete.

(iii) [104] Mixed-integer linear programming with fixed number of variables is in P .

(iv) [42] Linear complementarity problem is NP-complete.

(v) [16] Bilevel programming with $\mathcal{L} = \mathcal{F} = \emptyset$, i.e., continuous bilevel programming is NP-complete.

(vi) [105] Mixed-integer bilevel programming is Σ_2^P -complete.

2.2 Representability of problem classes

Recalling the definition of representability in Definition 8, one can readily see that the class of linear program-representable sets is set of all polyhedra. Indeed, any feasible set for a linear program is a polyhedron, and linear transforms of polyhedra are always polyhedra. However, the increase in modeling power by using integer variables is precisely quantified by the following result by Jeroslow and Lowe [92].

Theorem 15. [92] *Let S be a mixed-integer representable set. Then,*

$$S = \bigcup_{i=1}^k P^i + \text{int cone}(\{r^1, \dots, r^\ell\}) \quad (2.2)$$

where each P^i is a polytope, $r^1, \dots, r^\ell \in \mathbb{Q}^n$ and the operation $\text{int cone}(M)$ corresponds to all points obtained as a sum of non-negative integer multiples of elements in M .

The representability of linear complementarity problems and mixed-integer bilevel programs is precisely characterized as a part of the author's doctoral work. We postpone these results till [Chapter 6](#).

2.3 Applications

[Theorem 15](#) due to Jeroslow and Lowe [\[92\]](#) assures that one can optimize over a finite union of polytopes (and many other sets too) by using mixed-integer sets. Optimizing over such families of sets have importance in real life. In fact, in this section, we show that important real-life problems could be modeled using the paradigms of nonconvexity considered in this dissertation. In particular, [Chapter 4](#) is dedicated to describing the modeling of real-life problems as complementarity problems.

We refer the reader to Conforti et al. [\[43, Chapter 2\]](#), Cook et al. [\[46\]](#) and references therein for a fairly comprehensive list of problems which can be solved as an MIP. Facchinei and Pang [\[66, Section 1.4\]](#) gives a detailed set of problems that can be solved using complementarity constraints. We refer the reader to Bard [\[16\]](#), Dempe [\[52\]](#) for an overview of the applications of mixed-integer bilevel programming. We devote the rest of the section to motivate an application of complementarity problems which is extensively used during the author's doctoral work.

2.3.1 Games and equilibria

We first define a Nash equilibrium problem. We motivate this from economics where finitely many players *simultaneously* make decisions about the goods they produce or the services they offer.

Definition 16 (Nash equilibrium problem). *A Nash equilibrium problem is defined as a finite tuple of optimization problems (P^1, \dots, P^n) , where each P^i has the following form.*

$$\min_{x^i \in \mathbb{R}^{n_i}} \quad : \quad f^i(x^i; x^{-i})$$

subject to

$$g_j^i(x^i) \leq 0 \quad j = 1, \dots, k_i$$

where x^{-i} is defined as $(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n)$. We call P^i as the problem of the i^{th} player.

With the same terminology, f^i is the i^{th} player's objective and g_j^i are the constraints of i^{th} player.

$\bar{x} \in \mathbb{R}^{\sum_{i=1}^n n_i}$ is called a Nash equilibrium, if \bar{x} solves each of the optimization problems P^i .

An alternative way to look at this problem is by observing we have n optimization problems, each of which has a set of decision variables, and each problem's variables parameterizes other problems' objective functions. Observe that the definition implies that no player can *unilaterally* change her decision from Nash equilibrium and get a better objective. Further, note that if each f^i is a function *only* of x^i , i.e., if the players don't interact with each other, then we can trivially solve n individual optimization problems and we obtain the Nash equilibrium. However, the interactions between the individual problems require more sophisticated techniques to solve for Nash equilibrium.

We now state [Assumptions A1](#) to [A3](#) which hold throughout this dissertation.

Assumption A1 (Compact strategies). *For each i , $\{x : g_j^i(x) \leq 0, j = 1, \dots, k_i\}$ is a compact set.*

If [Assumption A1](#) does not hold, we can make simple games with no Nash equilibrium. For example, consider the game with two players where each player's objective is to minimize $x^i - x^{-i}$ where $x^i \in \mathbb{R}$. Both players' objectives are unbounded and there exists no Nash equilibrium.

Now we make assumptions on convexity and smoothness of f^i and g_j^i . These assumptions enable us to reformulate the problem of finding the Nash equilibrium as a complementarity problem.

The assumptions of convexity and smoothness mentioned below ensure that the problem of finding the Nash equilibrium can be posed as a complementarity problem.

Assumption A2 (Convexity). *(a) Each f^i is a convex function in x^i .*

(b) Each g_j^i is an affine function. In other words, each P^i is a convex optimization problem with affine constraints.

Here the assumption (b) ensures that certain so-called constraint qualifications and the so-called Slater's conditions always hold [119]. This assumption eliminates pathological cases where [Theorem 17](#) does not hold.

Assumption A3 (Smoothness). *Each f^i is twice continuously differentiable.*

If these functions are continuously differentiable, that is sufficient to pose the problem of finding the Nash equilibrium as a complementarity problem. However, we assume twice continuous differentiability, which is crucial for the approximation algorithm developed in [Chapter 4](#).

We now state the theorem about the well known “Karush-Kuhn-Tucker conditions” or “KKT conditions.” These are first-order necessary and sufficient conditions for optimality in a smooth convex optimization problem.

Theorem 17. [\[99, 132\]](#) *Consider the problem*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) && \text{subject to} \\ g_j(x) \leq \quad & 0 && \text{for } j = 1, \dots, m \end{aligned} \tag{2.4}$$

where f is a convex function and g_j are all affine functions. Then $\bar{x} \in \mathbb{R}^n$ solves the problem if and only if $\exists \bar{u} \in \mathbb{R}^m$ such that the following hold.

$$\frac{\partial f(\bar{x})}{\partial x_i} + \sum_{j=1}^m \bar{u}_j \frac{\partial g_j(\bar{x})}{\partial x_i} = 0 \quad \forall i \in [n] \quad (2.5a)$$

$$\bar{u}_j \geq 0 \quad \forall j \in [m] \quad (2.5b)$$

$$g_j(\bar{x}) \leq 0 \quad \forall j \in [m] \quad (2.5c)$$

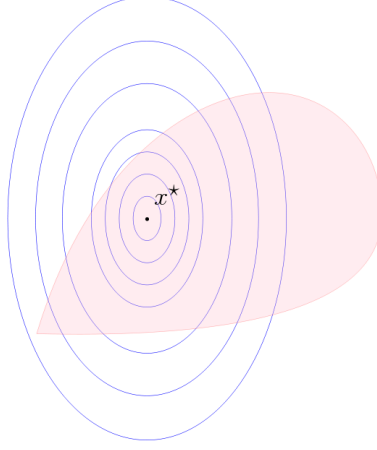
$$\bar{u}_j g_j(\bar{x}) = 0 \quad \forall j \in [m] \quad (2.5d)$$

We call [eq. \(2.5\)](#) as the KKT-conditions for the convex optimization problem. u_j is called the dual variable or the Lagrange multiplier corresponding to the constraint g_j .

Formally, the condition that each g_i has to be affine can be relaxed to just each g_i being convex. Then, the so-called constraint qualifications are needed to ensure that the “only if” part of the theorem holds and non-empty interior of the feasible set (otherwise called Slater’s condition) is needed to ensure that the “if” part of the theorem holds. Such conditions are satisfied trivially when g_j are all affine. We avoid the technicalities for brevity and refer curious readers to Facchinei and Pang [66], Nocedal and Wright [119], Ruszczyński [124] for a more detailed treatment of constraint qualifications and to Boyd and Vandenberghe [34], Nocedal and Wright [119] for Slater’s conditions.

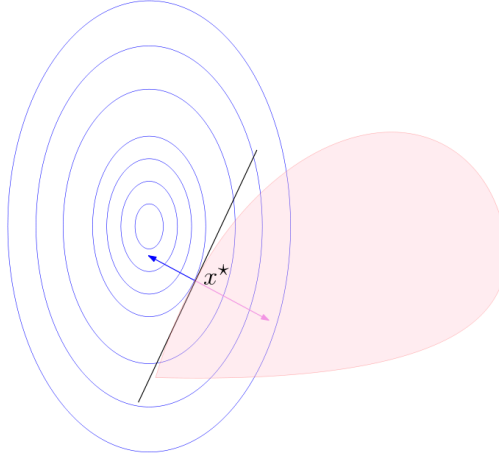
Observe that the condition in [eqs. \(2.5b\) to \(2.5d\)](#) is indeed the complementarity relation. This observation is used in [Theorem 18](#) to convert a non-cooperative game into a complementarity problem. The intuition behind these conditions are given in [Figure 2.1](#).

Theorem 18. [66, 77] *Let P be a non-cooperative game satisfying [Assumptions A1 to A3](#). Then there exists a nonlinear complementarity problem whose every solution is a Nash equilibrium of P . Further if each P^i has a convex quadratic objective and affine constraints, then we have a linear complementarity problem whose every solution is a Nash equilibrium of P .*



(a) Observe that the unconstrained optimum can already be strictly feasible ($g(x) < 0$) for the problem. In such cases, the gradient of the function defining the feasible region need not have any significant connection to the gradient of the objective. But the gradient of the objective is forced to be zero. This is ensured by

eq. (2.5a). Equation (2.5d) ensures $u = 0$ when $g(x) \neq 0$.



(b) Observe that the gradient of the function (blue arrow) is collinear with the gradient of the function defining the feasible region (pink arrow). This is ensured by eq. (2.5a) that $\nabla f = -u\nabla g$, in one constraint case. The fact that they point to opposite direction is ensured by eq. (2.5b), $u \geq 0$.

Figure 2.1: Intuition behind the KKT conditions: The blue circles are the contours of the objective function. The pink region is the feasible region and the point denoted by x^* is the optimal solution. Note that the optimal solution should be feasible, which is ensured by eq. (2.5c).

Proof. [66]

Write the KKT conditions of each problem P^i using dual variables u^i as follows.

$$\begin{aligned} \nabla_{x^i} f^i(x^i; x^{-i}) + u^{iT} \nabla_{x^i} g^i(x^i) &= 0 \\ g^i(x^i) &\leq 0 \\ u^i &\geq 0 \\ (u^i)^T g^i &= 0 \end{aligned}$$

Now using auxiliary variables, $\lambda_+^i, \lambda_-^i \in \mathbb{R}^{n_i}$, we can write the above as

$$\begin{aligned} 0 &\leq -\nabla_{x^i} f^i(x^i; x^{-i}) - u^{iT} \nabla_{x^i} g^i(x^i) && \perp && \lambda_-^i &\geq 0 \\ 0 &\leq \nabla_{x^i} f^i(x^i; x^{-i}) + u^{iT} \nabla_{x^i} g^i(x^i) && \perp && \lambda_+^i &\geq 0 \\ 0 &\leq g^i(x^i) && \perp && u^i &\geq 0 \end{aligned}$$

Clearly, this is a complementarity problem for each i . But even in case of multiple players, they can all be stacked vertically, and we still have a complementarity problem whose solution, after projecting out u^i, λ_+^i and λ_-^i , is simultaneously optimal to each of the n player's problem, which by definition is the Nash equilibrium. Further, by the construction, it is trivial to see that if f is quadratic, ∇f is linear. Now if all g^i are also affine, we have a *linear* complementarity problem. ■

Chapter 3

Mixed-integer programming

3.1 Introduction

In this chapter, we describe a well-studied branch of nonconvexity known as mixed-integer nonconvexity. This corresponds to the constraint that some of the variables should take integer values. As such, problems of this variety are ubiquitous in the fields of engineering and management. Some applications of integer programs are discussed in Conforti et al. [43, Chapter 2]. In this chapter, we discuss some of the algorithmic improvements provided to solve mixed-integer programs.

We organize the chapter as follows. We use the rest of this section to motivate some problems that can be posed as a mixed-integer linear program. Then we describe the standard branch-and-cut algorithm used to solve such problems. We then dedicate [Section 3.2](#) to motivate the theory behind cut-generating functions and some well-known results in that area. Finally, the author's work is detailed in [Section 3.3](#).

3.1.1 Mixed-integer programming

We recall the definition of a mixed-integer program.

Definition 10 (Mixed-integer program). Given $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $\mathcal{I} \subseteq [n]$, the mixed-integer program (MIP) is to find an element in the set

$$\{x \in \mathbb{R}^n : Ax \leq b, x_i \in \mathbb{Z} \text{ if } i \in \mathcal{I}\}$$

or to show that the set is empty.

As noted before, we say that the mixed-integer program is in *standard form*, if the question is about the non-emptiness of the set $\{x \in \mathbb{R}^n : Ax = b, x \geq 0, x_i \in \mathbb{Z} \text{ if } i \in \mathcal{I}\}$. Further if $\mathcal{I} = [n]$, we call the problem as a *pure integer program* and if $\mathcal{I} = \emptyset$, the problem is a linear programming problem. Finally, the family of these sets whose non-emptiness is being checked in a mixed-integer program would be referred to as *mixed-integer sets*.

Remark 19. We alternatively talk about mixed-integer (or pure integer or linear) optimization when we are interested in optimizing a *linear* function over the set whose non-emptiness is being checked. We also remind the reader about [Remark 5](#) where we noted that the feasibility version and the optimization version of the problem are equivalent.

3.1.2 Some source problems

In this subsection, we show some of the problems that can be posed as a mixed-integer program.

Knapsack problem

Suppose a person has a *knapsack* that can carry a weight W . Suppose there are n articles of value v_1, \dots, v_n that the person would like to carry in her knapsack. Let us say that each of the objects weighs w_1, \dots, w_n and the total weight that the person can carry should not exceed the capacity, W of the knapsack. Now, which articles should the person pick so as to maximize the value of the articles she carries in her knapsack?

This problem can be written concisely as the following mixed-integer program (refer [Remarks 5](#) and [19](#)).

$$\max_{x \in \mathbb{R}^n} \{v^T x : w^T x \leq W; 0 \leq x \leq 1; x_i \in \mathbb{Z} \text{ if } i \in [n]\} \quad (3.1)$$

where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. One can observe that the above problem is a pure integer program. The conditions $0 \leq x \leq 1$ enforces that each component of x is either a 0 or a 1. The solution to the original problem corresponds to carrying the objects i whose corresponding component in x is a 1 and not carrying the rest. The constraint $w^T x \leq W$ ensures that the total weight of the objects carried does not exceed W and $v^T x$ adds the values of objects that are carried, which is indeed maximized.

Chance-constrained linear program

A linear program, as stated in [Theorem 14](#) is an example of a convex optimization problem that can be solved efficiently in polynomial time. It has a variety of applications in its original form and its applicability gets enhanced as a chance-constrained linear program, which allows that certain conditions are satisfied with a probability, $1 - \varepsilon$. More formally, let us consider a bounded chance constrained program,

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad : \quad c^T x \quad (3.2a)$$

subject to

$$\mathbb{P}(Tx \geq \xi) \geq 1 - \varepsilon \quad (3.2b)$$

$$Ax = b \quad (3.2c)$$

$$x \geq l \quad (3.2d)$$

$$x \leq u \quad (3.2e)$$

where $T \in \mathbb{R}^{m \times n}$ is a known matrix, $1 \geq \varepsilon > 0$ is a known scalar and $\xi : \Omega \mapsto \mathbb{R}^m$ is a random variable with finite support. In particular, let us say that ξ takes values from ξ^1, \dots, ξ^ℓ with probabilities π_1, \dots, π_ℓ respectively. Indeed, $0 \leq \pi_i \leq 1$ for all $i \in [\ell]$ and $\sum_{i=1}^\ell \pi_i = 1$. These problems can be shown to be *NP*-complete [110], while their deterministic counter parts are polynomially solvable.

To solve these problems, Luedtke, Ahmed, and Nemhauser [110] propose a mixed-integer program reformulation for this problem.

$$\begin{aligned} \text{Minimize} \quad & c^T x \\ & \begin{matrix} x \in \mathbb{R}^n \\ y \in \mathbb{R}^m \\ z \in \{0,1\}^\ell \end{matrix} \end{aligned} \quad (3.3a)$$

subject to

$$Ax = b \quad (3.3b)$$

$$x \geq l \quad (3.3c)$$

$$x \leq u \quad (3.3d)$$

$$y = Tx \quad (3.3e)$$

$$y + M(1 - z_i)e \geq \xi_i \quad \forall i \in [\ell] \quad (3.3f)$$

$$\sum_{i=1}^n \pi_i z_i \geq \varepsilon \quad (3.3g)$$

where M is a large number and e is a vector of ones. Such a number M exists from the boundedness of the problem. Equation (3.3) is a mixed-integer program in variables $(x, y, z) \in \mathbb{R}^{n+m+\ell}$. For some i , if $z_i = 0$, then eq. (3.3f) is trivially satisfied since M is large. On the other hand, if $z_i = 1$,

then the constraint $Tx = y \geq \xi_i$ is enforced. Then, [eq. \(3.3g\)](#) ensures that the set of i for which $Tx \geq \xi_i$ is satisfied is sufficiently large that the chance constraint is satisfied.

Other problems:

Knapsack problem is a traditional example of a mixed-integer program. The fact that a chance-constrained optimization program can be rewritten as a mixed-integer program is known for not more than a decade. Besides these, boolean satisfiability problem, clique problem, stable set problem, problem of finding a Hamiltonian cycle in a graph, the travelling-salesman problem, vertex-cover problem, set packing problem, set covering problem, machine scheduling problem, open pit mining problem are all standard problems which can be reduced into a mixed-integer program.

3.1.3 Preliminary definitions and algorithms for mixed-integer programs

Definition 20 (Linear relaxation of an MIP). *Given the mixed-integer program of finding an element in the set $\{x : Ax \leq b, x_i \in \mathbb{Z} \text{ if } i \in \mathcal{I} \subseteq [n]\}$ or showing its emptiness, the linear relaxation of the problem is to find an element in the set $\max_{x \in \mathbb{R}^n} \{Ax \leq b\}$ or to show that this set is empty. We also call this as the LP relaxation of the problem.*

Remark 21. Note that the definition of linear relaxation can also be defined with the idea of optimizing a linear function, as described in [Remarks 5](#) and [19](#).

Definition 22 (Valid inequality). *Let $S \subseteq \mathbb{R}^n$. We say that $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ is a valid inequality for S if $\{x : \langle a, x \rangle \leq b\} \supseteq \text{conv}(S)$.*

Definition 23 (Separation oracle). *Let $S \subseteq \mathbb{R}^n$. Then $\Omega_S : \mathbb{R}^n \mapsto (\{\text{"Yes"}\} \cup (\mathbb{R}^n \times \mathbb{R}))$ is called a separation oracle for S , if $\Omega_S(x) = \text{"Yes"}$ if $x \in \text{conv}(S)$ and $\Omega_S(x) = (a, b)$ such that (a, b)*

is a valid inequality for S and $\langle a, x \rangle > b$ otherwise. We call the set $\{x \in \mathbb{R}^n : \langle a, x \rangle = b\}$ as the separating hyperplane returned by the oracle.

Theorem 24 (Grötschel et al. [85]). *Let $S \subseteq \mathbb{R}^n$ and let Ω_S be its separation oracle. Then for any $c \in \mathbb{R}^n$, $\max_{x \in S} c^T x$ can be solved with polynomially many calls to Ω_S and polynomially many elementary operations.*

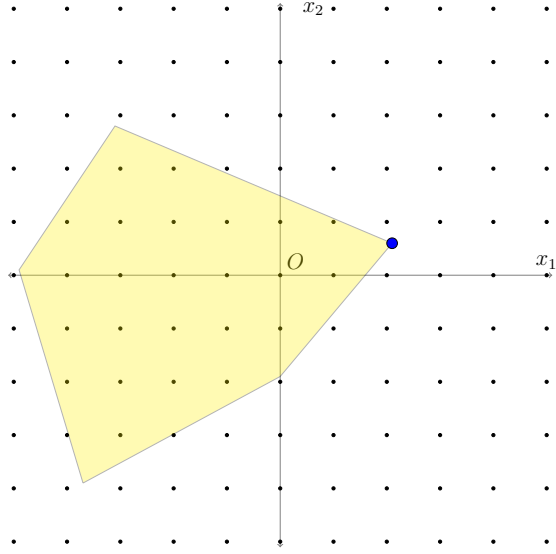
The above theorem was proved using the ellipsoidal algorithm provided by Khachiyan [97] to solve linear programs in polynomial time. For the case of linear programs, the separation oracle is defined by the set of constraints defining the linear program. Grötschel et al. [85] extends the result by saying that *any* optimization problem with a separation oracle can be solved using polynomially many calls to the oracle.

Lemma 25 ([126]). *Consider the linear program $\max_{x \in P} \{c^T x\}$ where P is a polyhedron. Assume that this problem has a finite solution. Then there exists $v \in P$, a vertex of the polyhedron such that v solves the linear program. We call such a v as a vertex solution of the linear programming problem.*

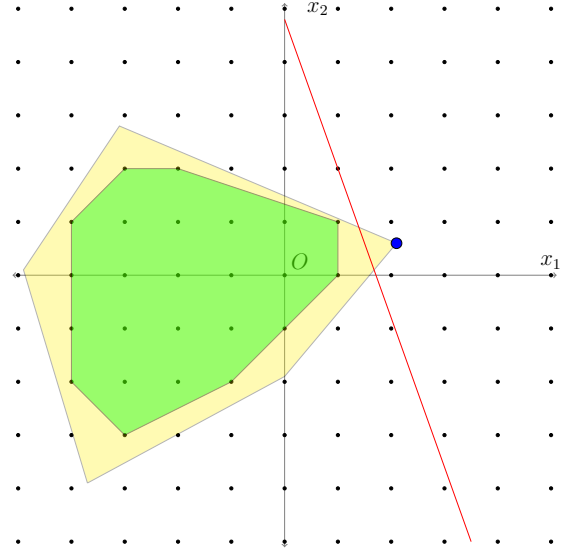
In fact, we have algorithms (for example, the Simplex algorithm [126]) that are guaranteed to return a vertex solution for a linear programming problem or a certificate that says that the problem is either infeasible or unbounded.

Cutting plane approach

Cutting plane methods were first introduced in Gomory [82] as a means to solve pure integer programs. Cutting plane methods iteratively generate valid inequalities for the mixed-integer set of interest, typically “cutting off” the solution to the LP relaxation of the problem.



(a) The shaded region is the polyhedron $P \subset \mathbb{R}^2$. We are interested in $\max_{x \in P \cap \mathbb{Z}^2} \{x_1\}$. The solution to the LP relaxation is shown as a blue dot.



(b) The convex hull of $P \cap \mathbb{Z}^2$ is shown in green. The cutting plane which cuts off the LP relaxation's solution but is valid for $\text{conv}(P \cap \mathbb{Z}^2)$ is shown in red.

Figure 3.1: Intuition behind cutting planes for MIP

Example 26 (Cutting plane). *Consider the problem,*

$$\max_{x,y,z \in \mathbb{Z}} \left\{ \begin{array}{lcl} x + y & \leq & 1 \\ x + y + z : & y + z & \leq 1 \\ & x + z & \leq 1 \end{array} \right\}$$

. One can verify that the solution to the LP relaxation of this problem is $x = 0.5, y = 0.5, z = 0.5$. Now to solve the original problem, one can sum the constraints $x + y \leq 1, y + z \leq 1, z + x \leq 1$ to get $2(x + y + z) \leq 3$ or $x + y + z \leq 1.5$. However, given the condition that each of x, y, z have to be integers, one can say that their sum has to be an integer. Thus the inequality $x + y + z \leq 1.5$ implied by the given set of constraints can be strengthened to $x + y + z \leq 1$, because of the integrality of the variables. Note that the solution to the LP relaxation violates this new constraint or in other

words, “cuts off” the LP relaxation’s solution. Such an inequality which is valid for the original problem but not to the LP relaxation’s solution is a *cutting plane*.

There are many methodical algorithms to generate cutting planes and their advent has greatly improved the speed of our algorithms for MIPs. We discuss some of the ideas behind generating cutting planes in greater detail in [Section 3.2](#).

Next, we describe the branch-and-cut algorithm to solve a mixed-integer program.

Branch-and-cut algorithm

The branch-and-cut algorithm is the most widely used algorithm for mixed-integer programs. The algorithm iteratively solves linear programming problems. If the solution to the linear program already satisfies the integer constraints in the MIP, then we are done. If not, we do one of the following - (i) add cutting planes that make the original LP solution infeasible and we solve another LP (ii) or we choose a variable whose integer constraints are not fulfilled and *branch* on that variable. This creates a tree of linear programs one of whose solutions either solves the original MIP or we identify that the mixed-integer set is empty (or infeasible). While in the worst case, this algorithm can end up solving exponentially many linear programs, in practice, this works very well and forms the basis for every commercial solver to solve MIPs. An outline of this algorithm is given in [Algorithm 1](#). A close variant of this algorithm is the *branch-and-bound* algorithm that always chooses to branch in [line 8](#) of [Algorithm 1](#). On the other hand, if the algorithm always chooses to cut in [line 8](#) of [Algorithm 1](#) is called the *cutting-plane algorithm*.

Algorithm 1 Branch-and-cut algorithm

Given $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $\mathcal{I} \subseteq [n]$, maximize $c^T x$ over the mixed-integer set defined by

A, b, \mathcal{I} . Initialize $\mathcal{L} = \{(A, b)\}$, $x^* = \emptyset$, $\underline{z} = -\infty$.

- 1: **while** $\mathcal{L} \neq \emptyset$ **do**
 - 2: Choose some $(A_i, b_i) \in \mathcal{L}$.
 - 3: Solve $\text{LP}_i : \max_x \{c^T x : A_i x \leq b_i\}$. Let x^i solve LP_i and z_i be the optimal objective value.
 If it is infeasible, set $x^i = 0$, $z_i = -\infty$.
 - 4: **if** $z_i \leq \underline{z}$ **then** Go to [line 2](#). **end if** ▷ Fathom node
 - 5: **if** x^i satisfies the integrality constraints **then** ▷ Update incumbent
 - 6: $x^* = x^i$ and $\underline{z} := z_i$. Go to [line 2](#).
 - 7: **end if**
 - 8: Choice \leftarrow “cut” or “branch” ▷ Choose to cut or to branch
 - 9: **if** Choice is “cut” **then**
 - 10: Add cutting planes to the constraints $A_i x \leq b_i$ generated from *some cutting plane procedure*. Go to [line 3](#).
 - 11: **else** ▷ Branch
 - 12: Find a component $j \in \mathcal{I}$ such that $x_j^i \notin \mathbb{Z}$.
 - 13: Define two new (A', b') and (A'', b'') such that both includes all rows of (A_i, b_i) . One of them has $x_j \leq \lfloor x_j^i \rfloor$ constraint and the other has $x_j \geq \lceil x_j^i \rceil$ constraint.
 - 14: Include (A', b') and (A'', b'') in \mathcal{L} . ▷ New branches
 - 15: **end if**
 - 16: **end while**
 - 17: Return x^* .
-

3.2 Cut generating functions

One of the well-studied areas of the [Algorithm 1](#) which is still an active area of research is to implement the cutting plane generation procedure mentioned in [line 10](#). Bixby and Rothberg [\[28\]](#), Bixby et al. [\[29\]](#) give an account of how the advent of cutting planes has increased the speed of MIP solvers by over 5000% in standardized test problem sets. In this section, we briefly deal with some ideas behind generating cutting planes for MIP.

Gomory [\[81\]](#) introduced the idea of obtaining a cutting plane, given the solution to the LP relaxation associated with the MIP. Suppose we are interested in generating a valid inequality for the (pure) integer points in $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, then the following can be done. Suppose we find $y \geq 0$ such that $y^T A \in \mathbb{Z}^n$ and $y^T b \in \mathbb{R} \setminus \mathbb{Z}$, observe that for any $x \in \mathbb{Z}^n$, $(y^T A)x \in \mathbb{Z}$. So, we can strengthen $y^T Ax \leq y^T b$ to $y^T Ax \leq \lfloor y^T b \rfloor$, which is a valid inequality for the set of integer points in P but not necessarily for P . This is the basis for the so-called *Chvátal-Gomory procedure* and is the motivation for the *Gomory's fractional cuts* described in Gomory [\[81, 82\]](#). Note that the cutting plane derived in [Example 26](#) is a Gomory's fractional cut.

Following the Chvátal-Gomory procedure to generate valid inequalities for MIP, a large number of procedures and algorithms came into practice for generating cutting planes. These different algorithms to generate cutting planes gave rise to different families of cutting planes like mixed-integer rounding (MIR) inequalities, split inequalities, Gomory mixed-integer (GMI) inequalities, cover inequalities, clique inequalities, lift-and-project inequalities and so on. Conforti et al. [\[43, Chapter 5 and 6\]](#) give an extensive survey of different families of cutting planes.

The idea of cut generating functions provides a unifying framework to describe a large family of cutting planes for MIPs. Before a formal definition of cut generating functions, we brief the concept of corner polyhedron next.

3.2.1 Corner polyhedron

Let us consider a mixed-integer set presented in the standard form.

$$Ax = d \tag{3.4a}$$

$$x \geq 0 \tag{3.4b}$$

$$x_i \in \mathbb{Z} \quad \forall i \in \mathcal{I} \tag{3.4c}$$

where $A \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^m$. Using the simplex method [119, 126], one can find the vertices of the polyhedron $P = \{x \in \mathbb{R}^n : Ax = d, x \geq 0\}$. Each vertex is associated with a so-called basic-index \mathcal{B} which is a subset of $[n]$ with exactly m elements, such that the submatrix formed by the columns of A in positions corresponding to the indices in \mathcal{B} is non-singular. This nonsingular matrix is called as the *basic matrix*. From the theory of simplex algorithm, one can find the coordinates of the vertex associated with \mathcal{B} as the point, whose \mathcal{B} coordinates are given by $B^{-1}d$ and all the other coordinates set to 0.

In other words, let $A = [B \ N]$ where $B \in \mathbb{R}^{m \times m}$ is the basic matrix and the remaining columns of A make the non-basic matrix $N \in \mathbb{R}^{m \times (n-m)}$. Then the constraint $Ax = d$ can be written as

$$Bx_B + Nx_N = d \tag{3.5}$$

where x_B corresponds to the \mathcal{B} coordinates of x and x_N corresponds to the remaining coordinates of x . Premultiplying by B^{-1} , the system [eq. \(3.4\)](#) can be rewritten as

$$x_B = B^{-1}d - B^{-1}Nx_N \tag{3.6a}$$

$$x_N \geq 0 \tag{3.6b}$$

$$x_B \geq 0 \tag{3.6c}$$

$$x_B, x_N \in \mathbb{Z} \quad (\text{if corresponding coordinate is in } \mathcal{I}) \tag{3.6d}$$

Gomory [83] introduced a useful relaxation of the above set of equations, i.e., to let go of the non-negativity of the basic variables that is imposed in [eq. \(3.6c\)](#). This popularly came to be known as the corner relaxation and the feasible region of this relaxation is known as the *corner polyhedron*.

The equations describing the corner polyhedron are given below

$$x_B = B^{-1}d - B^{-1}Nx_N \quad (3.7a)$$

$$x_N \geq 0 \quad (3.7b)$$

$$x_B, x_N \in \mathbb{Z} \quad (\text{if corresponding coordinate is in } \mathcal{I}) \quad (3.7c)$$

The corner polyhedron, is a relatively easier set for analysis compared to a general mixed-integer set. Most of the families of cutting planes are valid for the corner polyhedron. Next, we motivate some background material and results on subadditive functions which are central to the definition of cut generating functions in [Definition 28](#).

For the rest of the chapter, we will consider the mixed-integer set and its corner relaxation in the following canonical form.

Definition 27. *A mixed-integer set in canonical form for cut generating functions is*

$$X_S^+(R, P) = \left\{ (s, y) \in \mathbb{R}_+^k \times \mathbb{Z}_+^\ell : Rs + Py \in b + \mathbb{Z}_+^n \right\} \quad (3.8)$$

for some matrices $R \in \mathbb{R}^{m \times j}$, $P \in \mathbb{R}^{m \times \ell}$ for some $m \in \mathbb{Z}_+$. The corresponding corner relaxation in the canonical form is

$$X_S(R, P) = \left\{ (s, y) \in \mathbb{R}_+^k \times \mathbb{Z}_+^\ell : Rs + Py \in b + \mathbb{Z}^n \right\} \quad (3.9)$$

Note the equivalence between [Definition 27](#) and [eq. \(3.7\)](#). The solution of the linear program relaxation $B^{-1}d$ is equivalent to $-b$ in [Definition 27](#). R and S in [Definition 27](#) are submatrices of N in [eq. \(3.7\)](#) corresponding to continuous and integer nonbasic variables respectively. s, y in

[Definition 27](#) correspond to the continuous and integer nonbasic variables x_N in [eq. \(3.7\)](#). Also note that the origin (i.e., where all coordinates of s and z are zero) in this notation corresponds to the solution of the LP relaxation. This is because, the vertex solution of an LP has all its non basic entries as zeros.

Further, in practice, only a subset of the rows of the corner polyhedron defined in [eq. \(3.7\)](#) is used. But one can observe, it is still in the form motivated by [Definition 27](#) and cut generating functions can be used in this context.

Now we define functions that enable us to generate valid inequalities for $X_S(R, P)$. We call them as cut generating functions.

Definition 28 (Cut generating functions). *Let $\psi : \mathbb{R}^k \mapsto \mathbb{R}$ and $\pi : \mathbb{R}^\ell \mapsto \mathbb{R}$. Now (ψ, π) is called a cut generating function (CGF) pair or a valid pair, if*

$$\sum_{i=1}^k \psi(r^i) s_i + \sum_{i=1}^{\ell} \pi(p^i) y_i \geq 1 \quad (3.10)$$

is a valid inequality for $X_S(R, P)$ where r^i is the i -th column of R and p^i is the i -th column of P .

Note that, any valid inequality generated by a CGF is indeed a cutting plane, as it is valid for $X_S^+(R, P) \subseteq X_S(R, P)$, the mixed-integer set by definition and cuts off the LP relaxation's solution $(0, 0)$. Also note that the same pair (ψ, π) should give valid inequalities irrespective of k, ℓ, R and P to be a CGF pair. However, it can depend upon b , the negative of the LP relaxation's solution.

While [Definition 28](#) indeed defines a seemingly useful class of functions that can generate valid inequalities for a mixed-integer set, one might wonder if such a pair of functions exists. To answer that question, we define certain classes of functions and use them to construct CGFs.

Definition 29 (Gauge function). *Let $B \subseteq \mathbb{R}^n$ be a convex function such that $0 \in \text{int}(B)$. Then the gauge function of C , $\psi_B : \mathbb{R}^n \mapsto \mathbb{R}$ is defined as $\psi_B(x) := \inf \{t > 0 : \frac{x}{t} \in B\}$.*

Theorem 30 ([20, 56]). *Let B be a convex set such that $0 \in \text{int}(B)$ and $\text{int}(B) \cap (b + \mathbb{Z}^n) = \emptyset$. Then, (ψ_B, ψ_B) is a cut generating function pair.*

The above theorem gives a *constructive* proof on the existence of CGF. Now let us strengthen our requirements on cut generating functions that are guaranteed to give *strong* valid inequalities. From the valid inequalities that CGFs produce, one can observe that if the function value of ψ and π are smaller, the inequalities they provide are stronger. This leads us to the concept of minimal cut generating functions.

Definition 31 (Minimal cut generating functions). *A CGF pair (ψ, π) is called minimal if we have another CGF pair (ψ', π') such that $\psi'(s) \leq \psi(s)$ for all s and $\pi'(y) \leq \pi(y)$ for all y , then $\psi' = \psi$ and $\pi' = \pi$.*

Existence of minimal CGFs can be proved using Zorn's lemma.

Next, we define lifting, which helps us get stronger cut generating functions.

Definition 32 (Lifting). *Let $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ be a function such that (ψ, ψ) is a CGF. A function $\pi \leq \psi$ is a lifting of ψ if (ψ, π) is also a CGF.*

Theorem 33 (Trivial lifting). *Let (ψ, ψ) be a CGF. Then $(\psi, \tilde{\psi})$ is also a CGF where*

$$\tilde{\psi}(x) := \inf_{z \in \mathbb{Z}^n} \psi(x + z) \quad (3.11)$$

Here $\tilde{\psi}$ is called as the trivial lifting of ψ .

Definition 34 (Maximal $b + \mathbb{Z}^n$ -free convex set). *Let $B \subseteq \mathbb{R}^n$ be a convex set such that $0 \in \text{int}(B)$ and $\text{int}(B) \cap (b + \mathbb{Z}^n) = \emptyset$. B is called a maximal $b + \mathbb{Z}^n$ -free convex (MLC) set if for some convex set B' containing the origin such that $\text{int}(B') \cap (b + \mathbb{Z}^n) = \emptyset$, $B' \supseteq B$ implies $B' = B$.*

Remark 35. Given the notion that $b + \mathbb{Z}^n$ is the “lattice”, we alternatively refer to $b + \mathbb{Z}^n$ -free sets as lattice-free sets.

Theorem 36. *Let B be a maximal $b + \mathbb{Z}^n$ -free convex set. Then $\exists \pi : \mathbb{R}^n \mapsto \mathbb{R}$ such that (ψ_{B-b}, π) is a minimal CGF. Such a π is called a minimal lifting of ψ_B .*

While the above theorem is about the existence of a minimal lifting, it need not necessarily be unique. In fact, it can be shown for many $b + \mathbb{Z}^n$ -free convex sets that there are multiple minimal liftings. We use the definition below to identify some sets which have a unique minimal lifting.

With all the preliminary definitions and theorems here, we now discuss generating good cuts from CGFs efficiently.

3.3 Can cut generating functions be good and efficient?

In our opinion, there are two key obstacles to implementing such cut generating functions in state-of-the-art software:

1. There are too many (in fact, infinitely many) maximal $b + \mathbb{Z}^n$ free sets to choose from – this is the problem of *cut selection*,
2. For maximal $b + \mathbb{Z}^n$ free polyhedra with complicated combinatorial structure, the computation of the trivial lifting via [eq. \(3.11\)](#) is extremely challenging. Moreover, computing the values of other minimal liftings is even more elusive, with no formulas like [eq. \(3.11\)](#) available.

Thus, a central question in making cut generating function theory computationally viable, which also motivates the title of this section, is the following.

Can we find a “simple” subset of maximal $b + \mathbb{Z}^n$ free polyhedra such that two goals are simultaneously achieved:

- (i) provide guarantees that this “simple” subset of $b + \mathbb{Z}^n$ free sets gives a good approximation of the closure obtained by throwing in cuts from all possible maximal $b + \mathbb{Z}^n$ free sets, and
- (ii) cutting planes can be derived from them with relatively light computational overhead, either via trivial liftings or other lifting techniques.

Summary of results. The goal of this work is to make some progress in the above question. We achieve the following as a part of this work:

1. One may wonder if the trivial lifting function of the gauge can approximate any minimal lifting up to some factor. We show that there exist maximal $b + \mathbb{Z}^n$ free sets whose gauge functions have minimal liftings that are arbitrarily better than the trivial lifting (on some subset of vectors). More formally, we establish

Theorem 37. *Let n be any natural number and $\epsilon > 0$. There exists $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and a family \mathcal{F} of maximal $(b + \mathbb{Z}^n)$ -free sets such that for any $B \in \mathcal{F}$, there exists a minimal lifting π of ψ_B and $p \in \mathbb{R}^n$ satisfying $\frac{\pi(p)}{\psi_B(p)} < \epsilon$.*

2. Given an arbitrary maximal $b + \mathbb{Z}^n$ free set B , computing the trivial lifting using [eq. \(3.11\)](#) can be computationally very hard because it is equivalent to the notorious closest lattice vector problem in the algorithmic geometry of numbers literature [\[63\]](#). One could potentially write an integer linear program to solve it, but this somewhat defeats the purpose of cut generating functions: one would like to compute the coefficients much faster than solving complicated optimization problems like [eq. \(3.11\)](#) (and even harder IPs for general lifting). To overcome this issue, we isolate a particular family of maximal $b + \mathbb{Z}^n$ free sets that we call *generalized cross-polyhedra* (see [Definition 39](#) for a precise definition) and give an algorithm for computing

the trivial lifting function for any member of this family without using a high dimensional integer linear program. For this family, one needs $O(2^n)$ time to compute the gauge function because the $b + \mathbb{Z}^n$ free sets have 2^n facets, and one needs an additional $O(n2^n)$ time to compute the trivial lifting coefficient. *Recall that n corresponds to the number of rows used to generate the cuts.* This is much better complexity compared to solving [eq. \(3.11\)](#) using an integer program or the closest lattice vector (the latter will have to deal with an asymmetric, polyhedral gauge which is challenging). This is described in [Section 3.5](#); see [Algorithm 2](#). For a subfamily of generalized cross-polyhedra, both of these computations (gauge values and trivial lifting values) can actually be done in $O(n)$ time, which we exploit in our computational tests (see [Section 3.6.2](#)). We envision using this in software and computations in the regime $n \leq 15$.

3. From a theoretical perspective, we also show that our family of generalized cross-polyhedra can provide a finite approximation for the closure of cutting planes of the form

$$\sum_{i=1}^k \psi_B(r_i) s_i + \sum_{i=1}^{\ell} \widetilde{\psi}_B(p_i) y_i \geq 1.$$

More precisely, for any matrices $R \in \mathbb{R}^{n \times k}$, $P \in \mathbb{R}^{n \times \ell}$, and any maximal $b + \mathbb{Z}^n$ free set B , let $H_B(R, P) := \{(s, y) : \sum_{i=1}^k \psi_B(r_i) s_i + \sum_{i=1}^{\ell} \widetilde{\psi}_B(p_i) y_i \geq 1\}$. Let \mathcal{G}_b denote the set of all generalized cross-polyhedra (as applicable to $S = b + \mathbb{Z}^n$). Then, we have

Theorem 38. *Let $n \in \mathbb{N}$ and $b \in \mathbb{Q}^n \setminus \mathbb{Z}^n$. Define for any matrices R, P*

$$M(R, P) := \cap_{B \text{ maximal } b + \mathbb{Z}^n \text{ free set}} H_B(R, P)$$

$$G(R, P) := \cap_{B \in \mathcal{G}_b} H_B(R, P)$$

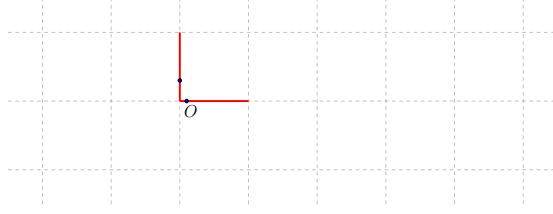
Then there exists a constant α depending only on n, b such that $M(R, P) \subseteq G(R, P) \subseteq \alpha M(R, P)$ for all matrices R, P .

Note that since $\psi_B, \tilde{\psi}_B \geq 0$, both $M(R, P)$ and $G(R, P)$ in [Theorem 38](#) are polyhedra of the blocking type, i.e., they are contained in the nonnegative orthant and have their recession cone is the nonnegative orthant. Thus, the relationship $G(R, P) \subseteq \alpha M(R, P)$ shows that one can “blow up” the closure $M(R, P)$ by a factor of α and contain $G(R, P)$. Equivalently, if we optimize any linear function over $G(R, P)$, the value will be an α approximation compared to optimizing the same linear function over $M(R, P)$.

4. We test our family of cutting planes on randomly generated mixed-integer linear programs, and also on the MIPLIB 3.0 set of problems. The short summary is that we seem to observe a tangible improvement with our cuts on the random instances, while no significant improvement in structured problems like MIPLIB 3.0 problems (except for a specific family). The random dataset consists of approx. 13000 instances and our observed improvement cannot be explained by random noise. More details are available in [Section 3.6](#).

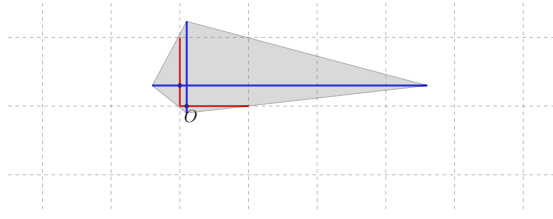
Our conclusion is that there is something to be gained from our cuts; however, it is unclear what kind of structures in MILP instances can be exploited by the proposed family in practice. It is also disappointing that these structures almost never appear in MIPLIB 3.0 problems. Nevertheless, we believe that future research will likely reveal when, in practice, our cuts will provide an improvement over current state-of-the-art methods. A very optimistic view that one can hold is that some future applications might contain models with the kind of structures which benefit from the proposed family of cuts.

The remainder of the chapter is dedicated to rigorously establishing the above results. [Section 3.4](#) formally introduces the class of generalized cross-polyhedra and [Theorem 38](#) is proved. [Section 3.5](#) then gives an algorithm for computing the trivial lifting for the family of generalized cross-polyhedra, which avoid solving integer linear programming problems or closest lattice vector

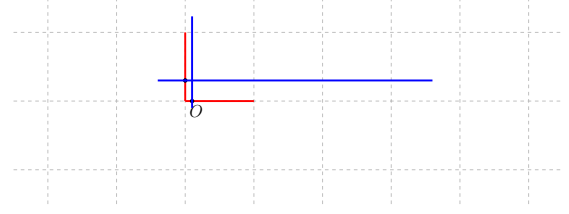


(a) The horizontal red line is the crosspolytope B and the vertical red line is the interval $I_{\lceil \gamma \rceil}$.

The points on the lines are c and γ .

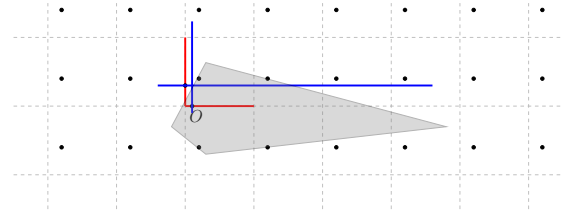


(c) The convex hull of the sets in [Figure 3.2b](#) gives G , the generalized cross-polytope.



(b) With $\mu = 0.25$, the horizontal blue line is $\left(\frac{1}{\mu}(B - c) + c\right) \times \{\gamma\}$ and the vertical line is

$$\{c\} \times \left(\frac{1}{1-\mu}(I_{\lceil \gamma \rceil} - \gamma) + \gamma\right).$$



(d) $b + G$ is the new $b + \mathbb{Z}^n$ free generalized cross-polytope.

Figure 3.2: Cross-polytope construction - The points $b + \mathbb{Z}^n$ are shown as black dots with \mathbb{Z}^n at the intersection of the grid.

problems for this purpose. [Section 3.6](#) gives the details of our computational testing. [Section 3.7](#) proves [Theorem 37](#).

3.4 Approximation by Generalized Cross Polyhedra

Definition 39. *[Generalized cross-polyhedra] We define the family of **generalized cross-polytopes** recursively. For $n = 1$, a generalized cross-polytope is simply any interval $I_a := [a, a+1]$, where $a \in \mathbb{Z}$. For $n \geq 2$, we consider any generalized cross-polytope $B \subseteq \mathbb{R}^{n-1}$, a point $c \in B$, $\gamma \in \mathbb{R}$, and $\mu \in (0, 1)$. A generalized cross-polytope in \mathbb{R}^n built out of B, c, γ, μ is defined as the convex hull of $\left(\frac{1}{\mu}(B - c) + c\right) \times \{\gamma\}$ and $\{c\} \times \left(\frac{1}{1-\mu}(I_{\lceil \gamma \rceil} - \gamma) + \gamma\right)$. The point $(c, \gamma) \in \mathbb{R}^n$ is called*

the **center** of the generalized cross-polytope.

A **generalized cross-polyhedron** is any set of the form $X \times \mathbb{R}^{n-m}$, where $X \subseteq \mathbb{R}^m$, $m < n$ is a generalized cross-polytope in \mathbb{R}^m .

The following theorem collects important facts about generalized cross-polyhedra that were established in Averkov and Basu [11], Basu and Paat [19] (where these sets were first defined and studied) and will be important for us below.

Theorem 40. *Let $G \subseteq \mathbb{R}^n$ be a generalized cross-polyhedron. The following are all true.*

- (i) *Let $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ such that $-b \in \text{int}(G)$ and let $S = b + \mathbb{Z}^n$. Then $b + G$ is a maximal S -free convex set. Moreover, using the values of c, γ and μ in the recursive construction, one can find normal vectors $a^1, \dots, a^{2^n} \in \mathbb{R}^n$ such that $b + G = \{x \in \mathbb{R}^n : a^i \cdot x \leq 1, i = 1, \dots, 2^n\}$.*
- (ii) *If G is a generalized cross-polytope, then there exists a unique $z \in \mathbb{Z}^n$ such that $z + [0, 1]^n \subseteq G \subseteq \cup_{j=1}^n ((z + [0, 1]^n) + \ell_j)$, where ℓ_j is the j -th coordinate axis obtained by setting all coordinates to 0 except coordinate j . Moreover, $z_j = \lfloor \gamma_j \rfloor$, where γ_j is the value used in the j -th stage in the recursive construction of G for $j = 1, \dots, n$.*

The main goal of this section is to establish the following result, which immediately implies [Theorem 38](#).

Theorem 41. *Let $b \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ such that the largest denominator in a coordinate of b is s . Let L be a $(b + \mathbb{Z}^n)$ -free set with $0 \in \text{int}(L)$. Then there exists a generalized cross-polyhedron G such that $B := b + G$ is a $(b + \mathbb{Z}^n)$ -free convex set such that $\left(\frac{1}{s^{4^{n-1} \text{Flt}(n)}}\right)^{n-1} L \subseteq B$.*

Let us quickly sketch why [Theorem 41](#) implies [Theorem 38](#).

Proof of [Theorem 38](#). We claim that $\alpha = \left(\frac{1}{s^{4^{n-1} \text{Flt}(n)}}\right)^{n-1}$ works. By a standard property of gauge functions, [Theorem 41](#) implies that for any maximal $b + \mathbb{Z}^n$ free set L , there exists a generalized

cross-polyhedron B such that $\psi_B \leq \alpha\psi_L$, consequently, by (3.11), $\tilde{\psi}_B \leq \alpha\tilde{\psi}_L$. Thus, $H_B(R, P) \subseteq \alpha H_L(R, P)$ and we are done. \blacksquare

The rest of this section is dedicated to proving Theorem 41. We need to first introduce some concepts and intermediate results, and the final proof of Theorem 41 is assembled at the very end of the section.

Definition 42 (Width function and lattice width). *For every nonempty subset $X \subset \mathbb{R}^n$, the width function $w(X, \circ) : \mathbb{R}^n \mapsto [0, \infty]$ of X is defined to be*

$$w(X, u) := \sup_{x \in X} x \cdot u - \inf_{x \in X} x \cdot u \quad (3.12)$$

The lattice width of X is defined as

$$w(X) := \inf_{u \in \mathbb{Z}^n \setminus \{0\}} w(X, u) \quad (3.13)$$

Definition 43 (Flatness). *The Flatness function is defined as*

$$\text{Flt}(n) := \sup \{w(B) : B \text{ is a } b + \mathbb{Z}^n \text{ free set in } \mathbb{R}^n\} \quad (3.14)$$

Theorem 44 (Flatness theorem [18]). $\text{Flt}(n) \leq n^{5/2}$ for all $n \in \mathbb{N}$.

Definition 45 (Truncated cones and pyramids). *Given an $n - 1$ -dimensional closed convex set $M \subset \mathbb{R}^n$, a vector $v \in \mathbb{R}^n$ such that $\text{aff}(v + M) \neq \text{aff}(M)$, and a scalar $\gamma \in \mathbb{R}_+$, we say that the set $T(M, v, \gamma) := \text{cl}(\text{conv}\{M \cup (\gamma M + v)\})$ is a truncated cone (any set that can be expressed in this form will be called a truncated cone).*

A truncated cone with $\gamma = 0$ is called a pyramid and is denoted $P(M, v)$. If M is a polyhedron, then $P(M, v)$ is a polyhedral pyramid. v is called the apex of $P(M, v)$ and M is called the base of $P(M, v)$. The height of a pyramid $P(M, v)$ is the distance of v from the affine hull of M .

When M is a hyperplane, the truncated cone is called a split.

Definition 46 (Simplex and Generalized Simplex). *A simplex is the convex hull of affinely independent points. Note that a simplex is also a pyramid. In fact, any facet of the simplex can be taken as the base, and the height of the simplex can be defined with respect to this base.*

A generalized simplex in \mathbb{R}^n is given by the Minkowski sum of a simplex Δ and a linear space X such that X and $\text{aff}(\Delta)$ are orthogonal to each other. Any facet of $\Delta + X$ is given by the Minkowski sum of a base of Δ and X . The height of the generalized simplex with respect to such a facet is defined as the height of Δ with respect to the corresponding base.

We first show that $b + \mathbb{Z}^n$ free generalized simplices are a good class of polyhedra to approximate other $b + \mathbb{Z}^n$ free convex bodies within a factor that depends only on the dimension. This result is a mild strengthening of Proposition 29 in Averkov et al. [12] and the proof here is very similar to the proof of that proposition.

Definition 47 (Unimodular matrix and unimodular transformations). *A matrix $U \in \mathbb{R}^{n \times n}$ is called unimodular if each element of the matrix $U_{ij} \in \mathbb{Z}$ and $\det(U) = \pm 1$. A linear transformation induced by a unimodular matrix is called unimodular transformation.*

In other words, unimodular transforms are invertible linear transforms U such that for any $z \in \mathbb{Z}^n$, $Uz \in \mathbb{Z}^n$. More informally they map \mathbb{Z}^n onto \mathbb{Z}^n .

Lemma 48. *Let $n \in \mathbb{N}$ and $b \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ such that the largest denominator in a coordinate of b is s . Let $S = b + \mathbb{Z}^n$. Then for any S -free set $L \subseteq \mathbb{R}^n$, there exists an S -free generalized simplex $B = \Delta + X$ (see Definition 46) such that $\frac{1}{s4^{n-1}\text{Flt}(n)}L \subseteq B$. Moreover, after a unimodular transformation, B has a facet parallel to $\{x \in \mathbb{R}^n : x_n = 0\}$, the height of B with respect to this facet is at most 1, and $X = \mathbb{R}^m \times \{0\}$ for some $m < n$.*

Proof. We proceed by induction on n . For $n = 1$, all S -free sets are contained in a lattice-free

$(b + \mathbb{Z} \text{ free})$ interval, so we can take B to be this interval. For $n \geq 2$, consider an arbitrary S -free set L . By [Theorem 44](#), $L' := \frac{1}{s4^{n-2} \text{Flt}(n)} L$ has lattice width at most $\frac{1}{s}$. Perform a unimodular transformation such that the lattice width is determined by the unit vector e^n and $b_n \in [0, 1]$.

If $b_n \neq 0$, then $b_n \in [1/s, 1-1/s]$, and therefore L' is contained in the split $\{x : b_n - 1 \leq x_n \leq b_n\}$.

We are done because all splits are generalized simplices and $\frac{1}{s4^{n-1} \text{Flt}(n)} L = \frac{1}{4} L' \subseteq L' \subseteq B := \{x : b_n - 1 \leq x_n \leq b_n\}$.

If $b_n = 0$, then $L \cap \{x : x_n = 0\}$ is an S' -free set in \mathbb{R}^{n-1} , where $S' = (b_1, \dots, b_{n-1}) + \mathbb{Z}^{n-1}$. Moreover, by the induction hypothesis applied to $L \cap \{x : x_n = 0\}$ and $L' \cap \{x : x_n = 0\}$ shows that there exists an S' -free generalized simplex $B' \subseteq \mathbb{R}^{n-1} \times \{0\}$ such that $L' \cap \{x : x_n = 0\} \subseteq B'$. Let B' be the intersection of halfspaces $H'_1, \dots, H'_k \subseteq \mathbb{R}^{n-1}$. By a separation argument between L' and $\text{cl}(\mathbb{R}^{n-1} \setminus H'_i) \times \{0\}$, one can find halfspaces $H_1, \dots, H_k \subseteq \mathbb{R}^n$ such that $H_i \cap (\mathbb{R}^{n-1} \times 0) = H'_i \times \{0\}$ and $L' \subseteq H_1 \cap \dots \cap H_k$ (this separation is possible because $0 \in \text{int}(L')$).

We now consider the set $P := H_1 \cap \dots \cap H_k \cap \{x : -1/s \leq x_n \leq 1/s\}$. By construction, $P \subseteq \mathbb{R}^n$ is S -free and $L' \subseteq P$ since L' has height at most $\frac{1}{s}$ and contains the origin. P is also a truncated cone given by $v = \frac{2}{s}e^n$ and $M = P \cap \{x : x_n = -1/s\}$ and some factor γ (see [Definition 45](#)), because B' is a generalized simplex. Without loss of generality, one can assume $\gamma \leq 1$ (otherwise, we change v to $-v$ and M to $P \cap \{x : x_n = 1\}$). By applying Lemma 25 (b) in Averkov et al. [\[12\]](#), one can obtain a generalized simplex B as the convex hull of some point $x \in P \cap \{x : x_n = \frac{1}{s}\}$ and M such that $\frac{1}{4}P \subseteq B \subseteq P$ (the hypothesis for Lemma 25 (b) in Averkov et al. [\[12\]](#) is satisfied because 0 can be expressed as the mid point of two points in $P \cap \{x : x_n = \frac{1}{s}\}$ and $P \cap \{x : x_n = -\frac{1}{s}\}$). Since $L' \subseteq P$, we have that $\frac{1}{s4^{n-1} \text{Flt}(n)} L = \frac{1}{4} L' \subseteq \frac{1}{4} P \subseteq B$. Since $B \subseteq P$, B is S -free. ■

Proof of [Theorem 41](#). We proceed by induction on n . If $n = 1$, then an S -free convex set is contained in an S -free interval, which is an S -free generalized cross-polyhedron, so we are done.

For $n \geq 2$, we consider two cases. Without loss of generality, we may assume $b_n \in [0, 1)$ (by translating everything by an integer vector).

By [Lemma 48](#), there exists an S -free generalized simplex $P = \Delta + X$ (see [Definition 46](#)) such that $\frac{1}{s4^{n-1}\text{Flt}(n)}L \subseteq P$. Moreover, after a unimodular transformation, P has a facet parallel to $\{x \in \mathbb{R}^n : x_n = 0\}$ and the height of P with respect to this facet is at most 1. Moreover, X can be assumed to be $\mathbb{R}^m \times \{0\}$ for some $m < n$. Thus, by projecting on to the last $n - m$ coordinates, we may assume that P is a simplex with a facet parallel to $\{x \in \mathbb{R}^n : x_n = 0\}$.

If $b_n \neq 0$, then $b_n \in [1/s, 1 - 1/s]$. Moreover, $\frac{1}{s}P$ has height at most $\frac{1}{s}$, and therefore it is contained in the maximal S -free split $\{x : b_n - 1 \leq x_n \leq b_n\}$. We are done because all maximal S -free splits are generalized cross-polyhedra and $\left(\frac{1}{s4^{n-1}\text{Flt}(n)}\right)^{n-1}L \subseteq \frac{1}{s}P \subseteq B := \{x : b_n - 1 \leq x_n \leq b_n\}$.

If $b_n = 0$, then by the induction hypothesis, there exists a translated generalized cross-polyhedron $B' \subseteq \mathbb{R}^{n-1} \times \{0\}$ such that $\left(\frac{1}{s4^{n-2}\text{Flt}(n-1)}\right)^{n-2}(P \cap \{x : x_n = 0\}) \subseteq B'$. Let v be the vertex of P with positive v_n coordinate. Since the height of P is at most 1, the height of $\left(\frac{1}{s4^{n-2}\text{Flt}(n-1)}\right)^{n-2}P$ is also at most 1. Let the facet F of $\left(\frac{1}{s4^{n-2}\text{Flt}(n-1)}\right)^{n-2}P$ parallel to $\{x \in \mathbb{R}^n : x_n = 0\}$ be contained in the hyperplane $\{x \in \mathbb{R}^n : x_n = \lambda\}$, where $-1 < \lambda < 0$ since P has height at most 1 with respect to this facet. Moreover, we may assume that after a unimodular transformation, the projection of v on to $\mathbb{R}^{n-1} \times \{0\}$ lies in B' , because the points from S on the boundary of B' form a lattice hypercube in \mathbb{R}^{n-1} by [Theorem 40\(ii\)](#). Let this projected vertex be $c \in \mathbb{R}^{n-1}$. Let $\mu = 1 - |\lambda|$ and $\gamma = \lambda$. Create the generalized cross-polyhedron B from B', c, μ, γ in \mathbb{R}^n as described in [Definition 39](#). By the choice of μ and γ and the fact that P has height at most 1, $v \in B$.

We also claim that $F \subseteq \left(\frac{1}{\mu}(B' - c) + c\right) \times \{\gamma\} \subseteq B$. Indeed, observe that

$$F - (c, \lambda) \subseteq \frac{1}{\mu} \left(\left(\left(\frac{1}{s4^{n-2}\text{Flt}(n-1)} \right)^{n-2} P \cap \{x \in \mathbb{R}^n : x_n = 0\} \right) - (c, 0) \right).$$

Since $\left(\frac{1}{s4^{n-2} \text{Flt}(n-1)}\right)^{n-2} (P \cap \{x : x_n = 0\}) \subseteq B'$, we have $F \subseteq \left(\frac{1}{\mu}(B' - c) + c\right) \times \{\gamma\}$.

Thus, we have that $\left(\frac{1}{s4^{n-2} \text{Flt}(n-1)}\right)^{n-2} P \subseteq B$ since $v \in B$ and $F \subseteq B$. Combining with $\frac{1}{s4^{n-1} \text{Flt}(n)} L \subseteq P$, we obtain that

$$\left(\frac{1}{s4^{n-1} \text{Flt}(n)}\right)^{n-1} L \subseteq \left(\left(\frac{1}{s4^{n-2} \text{Flt}(n-1)}\right)^{n-2}\right) \frac{1}{s4^{n-1} \text{Flt}(n)} L \subseteq B$$

■

3.5 Algorithms for trivial lifting in generalized cross-polyhedra

The key fact that we utilize in designing an algorithm to compute the trivial liftings of generalized cross-polyhedra is the following. By the results in Averkov and Basu [11], Basu and Paat [19], generalized cross-polyhedra have the so-called *covering property*. The essential implications of this for our purposes are distilled into the following theorem.

Theorem 49. *Let $G \subseteq \mathbb{R}^m$ be any generalized cross-polytope and let $b \in \mathbb{R}^m \setminus \mathbb{Z}^m$ such that $-b \in \text{int}(G)$. There is a subset $T \subseteq G$ such that $T + \mathbb{Z}^m = \mathbb{R}^m$ and for any $p \in \mathbb{R}^m$, there exists $\tilde{p} \in b + T$ such that $\tilde{p} \in p + \mathbb{Z}^m$ and $\widetilde{\psi_{b+G}}(p) = \psi_{b+G}(\tilde{p})$.*

The region T is called the covering region and the implication of Theorem 49 is that the family of generalized cross-polytopes have the *covering property*.

Thus, for any generalized cross-polyhedron $G \subseteq \mathbb{R}^m$ and $p \in \mathbb{R}^m$, if one can find the \tilde{p} in Theorem 49, then one can compute the trivial lifting coefficient $\widetilde{\psi_{b+G}}(p)$ by simply computing the gauge function value $\psi_{b+G}(\tilde{p})$. The gauge function can be computed by simple evaluating the 2^m inner products in the formula $\psi_{b+G}(r) = \max_{i=1}^{2^m} a^i \cdot r$, where $a^i, i = 1, \dots, 2^m$ are the normal vectors as per Theorem 40(i).

Thus, the problem boils down to finding \tilde{p} from [Theorem 49](#), for any $p \in \mathbb{R}^m$. Here, one uses property (ii) in [Theorem 40](#). This property guarantees that given a generalized cross-polytope $G \subseteq \mathbb{R}^m$, there exists $\bar{z} \in \mathbb{Z}^n$ that can be explicitly computed using the γ values used in the recursive construction, such that $T \subseteq G \subseteq \cup_{j=1}^m ((\bar{z} + [0, 1]^m) + \ell_j)$, where ℓ_j is the j -th coordinate axis obtained by setting all coordinates to 0 except coordinate j . Now, for any $p \in \mathbb{R}^m$, one can first find the (unique) translate $\hat{p} \in p + \mathbb{Z}^n$ such that $\hat{p} \in b + \bar{z} + [0, 1]^m$ (this can be done since b and z are explicitly known), and then \tilde{p} in [Theorem 49](#) must be of the form $\hat{p} + Me^j$, where $M \in \mathbb{Z}$ and e^j is one of the standard unit vectors in \mathbb{R}^m . Thus,

$$\tilde{\psi}_{b+G}(p) = \min_{\substack{j \in \{1, \dots, m\}, \\ M \in \mathbb{Z}}} \psi_{b+G}(\hat{p} + Me^j).$$

For a fixed $j \in \{1, \dots, m\}$, this is a one dimensional convex minimization problem over the integers $M \in \mathbb{Z}$ for the piecewise linear convex function $\phi_j(\lambda) = \psi_{b+G}(\hat{p} + \lambda e^j) = \max_{i=1}^{2^m} a^i \cdot (\hat{p} + \lambda e^j)$. Such a problem can be solved by simply sorting the slopes of the piecewise linear function (which are simply $a_j^1, \dots, a_j^{2^n}$), and finding the point $\bar{\lambda}$ where the slope changes sign. Then either $\phi_j(\lceil \bar{\lambda} \rceil)$ or $\phi_j(\lfloor \bar{\lambda} \rfloor)$ minimizes ϕ_j . Taking the minimum over $j = 1, \dots, m$ gives us the trivial lifting value for p .

One observes that this entire procedure takes $O(m2^m)$. While this was described only for generalized cross-polytopes, generalized cross-polyhedra of the form $G \times \mathbb{R}^{n-m}$ pose no additional issues: one simply projects out the $n - m$ extra dimensions.

We give a formal description of the algorithm below in [Algorithm 2](#). We assume access to procedures $\text{GETNORMAL}(G, b)$ and $\text{GAUGE}(G, b, x)$. $\text{GETNORMAL}(G, b)$ takes as input a generalized cross-polytope G and b such that $-b \in \text{int}(G)$, and returns the list of normals $\{a^1, \dots, a^{2^n}\}$ such that $b + G = \{x \in \mathbb{R}^n : a^i \cdot x \leq 1, \ i = 1, \dots, 2^n\}$ (property (i) in [Theorem 40](#)). $\text{GAUGE}(G, b, r)$ takes as input a generalized cross-polytope G and b such that $-b \in \text{int}(G)$ and a

vector r , and returns $\psi_{b+G}(r)$ (given the normals from $\text{GETNORMAL}(G, b)$, one simply computes the 2^n inner products $a^i \cdot r$ and returns the maximum).

Algorithm 2 Trivial lifting of a generalized cross-polytope

Input: Generalized cross-polytope $G \subseteq \mathbb{R}^n$, $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ such that $-b \in \text{int}(G)$. $p \in \mathbb{R}^n$ where the lifting is to be evaluated.

Output: $\widetilde{\psi_{b+G}}(p)$

```

1: function CROSSPOLYLIFT( $G, \mathbf{b}, \mathbf{x}$ )

2:   Set  $\bar{z} \in \mathbb{R}^n$  using parameters of  $G$  as given in property (ii) in Theorem 40.

3:   Compute unique  $\hat{p} \in (p + \mathbb{Z}^n) \cap \mathbf{b} + \bar{z} + [0, 1]^n$ .

4:   Let  $\mathcal{N} = \text{GETNORMAL}(G, b)$  be the set of normals.

5:   for Each coordinate  $j$  from 1 to  $n$  do

6:       Find  $a^- \in \arg \max_{a \in \mathcal{N}} \{a_j : a_j \leq 0\}$  where  $a_j$  denotes the  $j$ -th coordinate of  $a \in \mathcal{N}$ .

       Break ties by picking the one with maximum  $a \cdot \hat{p}$ .

7:       Find  $a^+ \in \arg \min_{a \in \mathcal{N}} \{a_j : a_j > 0\}$  where  $a_j$  denotes the  $j$ -th coordinate of  $a \in \mathcal{N}$ .

       Break ties by picking the one with maximum  $a \cdot \hat{p}$ .

8:        $\bar{\lambda} \leftarrow \frac{a^+ \cdot \hat{p} - a^- \cdot \hat{p}}{a^- - a^+}$ .

9:        $m_j \leftarrow \min\{a^+ \cdot \hat{p} + \lceil \bar{\lambda} \rceil a_j^+, a^- \cdot \hat{p} + \lfloor \bar{\lambda} \rfloor a_j^-\}$ .

10:   end for

11:   return  $\min\{1, m_1, \dots, m_j\}$ .

12: end function

```

3.6 Computational Experiments and Results

In this section we give results from a set of computational experiments comparing the cuts described in this dissertation against Gomory’s Mixed Integer (GMI) cuts, and also CPLEX computations at the root node. In the next two subsections, we describe the data generation procedure and the cut generation procedure we use, respectively. Following that, we summarize our results.

3.6.1 Data generation

Most of the tests we mention in this section were performed on randomly generated data. We write all our test problems in the canonical form

$$\min_{x \in \mathbb{R}^d} \{c^T x : Ax = b; x \geq 0; i \in \mathcal{I} \implies x_i \in \mathbb{Z}\} \quad (3.15)$$

where $A \in \mathbb{R}^{k \times d}$, $b \in \mathbb{R}^k$, $c \in \mathbb{R}^d$ and $\mathcal{I} \subseteq \{1, 2, \dots, n\}$.

We generated roughly 12,000 problems in the following fashion.

- Each problem can be pure integer or mixed-integer. For mixed-integer programs, we decide if each variable is discrete or continuous randomly with equal probability.
- Each problem can have the data for A , b and c as matrices with either integer data or rational data represented upto 8 decimal places.
- The size of each problem varies from $(k, d) \in \{(10i, 25i) : i \in \{1, 2, \dots, 10\}\}$.
- There are roughly 300 realizations of each type of problem.

This leads to $2 \times 2 \times 10 \times 300$ (roughly) $\approx 12,000$ problems in all. This number is not precise as some random problems where infeasibility or unboundedness were discovered in the LP relaxation

were ignored. Below we present the results for these approximately 12,000 problems as a whole and also the performance of our methods in various subsets of these instances.

3.6.2 Cut generation

We consider three types of cuts in these computational tests - Gomory's mixed-integer (GMI) cuts, X-cuts and GX-cuts. GMI cuts are single row cuts obtained from standard splits Conforti et al. [43, Eqn 5.31]. GX-cuts are cuts obtained from certain structured generalized cross-polytopes defined in Definition 39. X-cuts are obtained from a special case of generalized cross-polytopes, where the center (c, γ) coincides with the origin. It should be noted that the GMIs are indeed a special case of X-cuts because they can be viewed as cuts obtained from $b + \mathbb{Z}^n$ free intervals or one-dimensional generalized cross-polytopes whose center coincide with the origin. In this section, we call such cross-polytopes as *regular cross-polytopes*. This motivates the set inclusions shown in Figure 3.5. The motivation behind classifying a special family of cross-polytopes with centers coinciding with the origin is the algorithmic efficiency they provide. Because of the special structure in these polytopes, the gauges and hence the cuts can be computed much faster than what we can do for an arbitrary generalized cross-polytope (comparing with the algorithms in Section 3.5). In particular, the gauge and the trivial lifting can both be computed in $O(n)$ time, as opposed to $O(2^n)$ and $O(n2^n)$ respectively for the general case (see Section 3.5), where n is the dimension of the generalized cross-polytopes or equivalently, the number of rows of the simplex tableaux used to generate the cut.

The family of generalized cross-polytopes that we consider can be parameterized by a vector $\mu \in (0, 1)^n$ and another vector in $f \in \mathbb{R}^n$. This vector consists of the values μ_i used in each stage of construction of the cross-polytope, after appropriate normalization (see Definition 39).

This actually forces $\sum_{i=1}^n \mu_i = 1$. The vector f corresponds to the center of the generalized cross-polytope; the coordinates of f give the coordinates of c and γ in the iterated construction of [Definition 39](#). Both the parameters μ and f show up in [Algorithm 3](#). The regular cross-polytopes are obtained by setting $f = \mathbf{0}$ in the above construction; thus, they are parameterized by only the vector $\mu \in (0, 1)^n$. As long as $\sum_{i=1}^n \mu_i = 1$, there exists a one-to-one map between such vectors and the set of regular cross-polytopes in \mathbb{R}^n .

3.6.3 Comparison procedure

In each of the problems, the benchmark for comparison was an aggressive addition of GMI cuts. The procedure used for comparison is mentioned in [Algorithm 3](#). We would like to emphasize that X-cuts and GX-cuts are an infinite family of cuts, unlike the GMI cuts. However, we add only finitely many cuts from this infinite family.

In all the computational tests in this dissertation, these cuts are randomly generated without looking into any systematic selection of rows or μ . However, to improve the performance from a completely random selection, we generate ℓ batches of k cuts and only keep the best set of k cuts. We lay out our testing procedure in detail in [Algorithm 3](#).

For the set of 12,000 problems, X-cuts and GX-cuts were generated with $N = 2, 5$, and 10 rows. For GX-cuts, the number q of rows to be picked whose corresponding basics violate integrality constraints was chosen to be 1. This was found to be an ideal choice under some basic computational tests with small sample size, where cuts with different values of q were compared. Also, a qualitative motivation behind choosing $q = 1$ is as follows: GMI cuts use the information only from those rows where integrality constraints on the corresponding basic variables are violated. To beat GMI, it is conceivably more useful to use information not already available for GMI cuts, and hence to look

at rows where the integrality constraint on the corresponding basic variable is not violated.

3.6.4 Results

A typical measure used to compute the performance of cuts is *gap closed* which is given by $\frac{\text{cut} - \text{LP}}{\text{IP} - \text{LP}}$. However, the IP optimal value IP could be expensive to compute on our instances. So we use a different metric, which compares the performance of the best cut we have, against that of GMI cuts. Thus we define

$$\beta = \frac{\text{Best} - \text{GMI}}{\text{GMI} - \text{LP}}, \quad (3.16)$$

which tries to measure the *improvement* over GMI cuts using the new cuts.

The testing procedure mentioned in [Algorithm 3](#) was run with the values of $k = \ell = 5$. The results hence obtained are mentioned in [Table 3.1](#). Besides this table, we present some interesting observations from our computational testing.

1. In mixed-integer programs, we have $\beta \geq 10\%$ in 648 cases (which is 9.53% of the set of mixed-integer programs). In pure-integer programs, we have $\beta \geq 5\%$ in 320 cases (which is 4.7% of the set of pure-integer programs). A conclusion from this could be that the family of cuts we are suggesting in this dissertation works best when we have a good mix of integer and continuous variables. We would like to remind the reader that in the mixed-integer examples we considered, roughly half the variables were continuous, due to a random choice between the presence or absence of integrality constraint for each variable.
2. We also did some comparisons between $N = 2, 5, 10$ row cuts. In particular, let us define β_2, β_5 and β_{10} as the values of β with $N = 2, 5, 10$ respectively. Among the 13,604 cases, only in 265 cases we found $\beta_5 > \beta_2$ or $\beta_{10} > \beta_2$ (the inequalities are considered strictly here).

Table 3.1: Results

| Filter | Number of problems | Cases where $\mathbf{GMI} < \mathbf{Best}$ | Average of β | Average of β when GMI is beaten |
|---|--------------------|--|--------------------|---------------------------------------|
| None (All problems) | 13604 | 6538 (48.06%) | 2.00% | 4.15% |
| Rational Data | 6600 | 3213 (48.68%) | 2.11% | 4.23% |
| Integer Data | 7004 | 3325 (47.47%) | 1.90% | 3.80% |
| Pure Integer programs | 6802 | 2189 (32.18%) | 0.69% | 2.146% |
| Mixed Integer programs | 6802 | 4376 (64.33%) | 3.32% | 5.159% |
| Rational Data Pure Integer programs | 3300 | 1078 (32.67%) | 0.75% | 2.306% |
| Rational Data Mixed Integer programs | 3300 | 2135 (64.70%) | 3.48% | 5.376% |
| Integer Data Pure Integer programs | 3502 | 1111 (31.52%) | 0.63% | 1.996% |
| Integer Data Mixed Integer programs | 3502 | 2241 (63.42%) | 3.17% | 4.95% |

In 264 of these cases, $\max\{\beta_5, \beta_{10}\} > \mathbf{GMI}$ (the inequality is strict here). In these 265 cases, 62 were pure-integer programs and GMI was beaten in all 62 problems. The other 203 cases were mixed-integer programs. GMI was beaten in 202 of these problems.

We conclude that when cuts derived from higher dimensional cross-polytopes dominate cut obtained from lower dimensional cross-polytopes, then the cuts from the higher dimensional cross-polytopes dominate GMI cuts as well. In other words, if we find a good cut from a high dimensional cross-polytope, then we have a very useful cut in the sense that it adds significant

value over GMI cuts.

3. Another test was done with increasing the number k which corresponds to the number of GX cuts added, from a constant 10 to half the number of GMI cuts in the problem (recall that for the results reported in [Table 3.1](#), $k = 5$). Integer data was used in this, and this test was performed in a smaller randomly generated sample of size 810. In pure integer cases, we beat GMI in about 25% cases and in mixed-integer programs, we beat GMI in 61% cases. The value of β is comparable to [Table 3.1](#) in both cases. But the lack of significant improvement suggests the following. The performance of cross-polytope based cuts is determined more by the problem instance characteristics, rather than the choice of cuts. If these cuts work well for a problem, then it should be reasonably easy to find a good cut.
4. Further there were 4 problems, all mixed-integer, with $\beta > 100\%$ implying that there could be a set of problems on whom a very good choice of rows and μ could give quite a significant improvement.
5. As far as the time taken to run these instances goes, for the number of rows considered in this test, most of the time is typically spent in solving the LP relaxation after addition of cuts, accessing the simplex tableaux to generate the cut etc., rather than actually computing the cut.

3.6.5 Performance in MIPLIB 3.0

Our testing with the new cuts discussed in this dissertation had meagre to no improvement in most of MIPLIB problems. Apart from the type of test mentioned in [Algorithm 3](#) above, we performed the following test motivated by Espinoza [\[65\]](#). We ran the MIPLIB problem on CPLEX 12.7.1, stopping after all root node calculations before any branching begins (CPLEX typically adds several

rounds of cuts at the root node itself). We keep count of the number of cuts added by CPLEX. Now we allow up to 10 times the number of cuts added by CPLEX, iteratively solving the LP relaxation after the addition of each cut. In each round, the cut that gives the best β among twenty-five randomly generated cut is added. We count the number of cuts we had to add and hence the number of rounds of LP we solve, to obtain an objective value as good as CPLEX. However, in almost all cases adding even ten times as many cuts as CPLEX did, did not give us the objective value improvement given by CPLEX.

Tests along the line of [Algorithm 3](#) were also not promising. The only set of exceptions is the **enlight** set of problems in MIPLIB 3.0. These are problems coming from the Enlight combinatorial game. The X-cuts did not show any improvement over GMI cuts. The performance of the GX-cuts are shown below in [Table 3.2](#). It can be seen from [Table 3.2](#) that the performance of GX cuts increases with the number of rows used.

Table 3.2: Performance on Enlight problems. The numbers reported are the optimal values of the LP after the corresponding cuts have been added (they are minimization problems).

| Problem | LP | GMI | 2 row GX | 5 row GX | 10 row GX | IP |
|----------------|-----------|------------|-----------------|-----------------|------------------|-----------|
| enlight9 | 0 | 1 | 1.1902 | 1.4501 | 1.9810 | INF |
| enlight13 | 0 | 1 | 1.1815 | 1.5410 | 1.9704 | 71 |
| enlight14 | 0 | 1 | 1.1877 | 1.5051 | 1.9195 | INF |
| enlight15 | 0 | 1 | 1.2001 | 1.4712 | 1.8991 | 69 |
| enlight16 | 0 | 1 | 1.1931 | 1.4934 | 1.8766 | INF |

3.7 Limitation of the trivial lifting: Proof of Theorem 37

In this section, we show that for a general $b + \mathbb{Z}^n$ free set, the trivial lifting can be arbitrarily bad compared to a minimal lifting. We first show that for $n = 2$, there exist $b \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ such that one can construct maximal $(b + \mathbb{Z}^2)$ -free triangles with the desired property showing that the trivial lifting of its gauge function can be arbitrarily worse than a minimal lifting.

Example in 2 dimensions: Consider the sequence of Type 3 Maximal $b + \mathbb{Z}^n$ free triangles with $b = (-0.5, -0.5)$ given by the equations

$$20x - y + 10.5 = 0 \quad (3.17a)$$

$$\alpha_i x + y + \frac{1 - \alpha_i}{2} = 0 \quad (3.17b)$$

$$-\beta_i x + y + \frac{1 + \beta_i}{2} = 0 \quad (3.17c)$$

with $\alpha_i = 1 + \frac{1}{i}$ and $\beta_i = \frac{1}{i}$. Let us call the sequence of triangles as T_i . The triangle T_1 is shown in Fig. 3.6.

For all i , the point $p = (0.25, 0)$ is located outside the region $T_i + \mathbb{Z}^n$. So clearly for all i , the trivial lifting evaluated at p is at least 1. However, let us consider the minimum possible value any lifting could take at p . This is given by (see Dey and Wolsey [56, Section 7], Basu et al. [22]):

$$\pi_{\min}(p) = \sup_{\substack{z \in \mathbb{Z}^n \\ w \in \mathbb{R}^n \\ w + Np \in b + \mathbb{Z}^n}} \frac{1 - \psi_{T_i}(w)}{N} \quad (3.18)$$

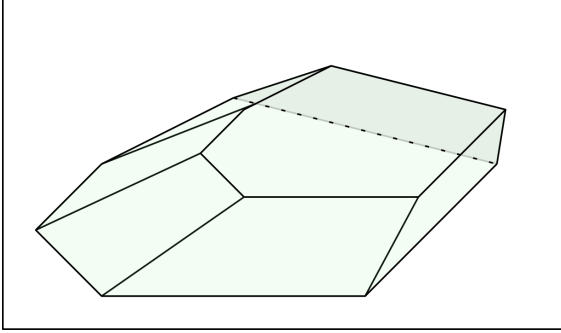
$$= \sup_{\substack{N \in \mathbb{N} \\ z \in \mathbb{Z}^n}} \frac{1 - \psi_{T_i}(b - Np + z)}{N} \quad (3.19)$$

$$= \sup_{N \in \mathbb{N}} \frac{1 - \inf_{z \in \mathbb{Z}^n} \psi_{T_i}(b - Np + z)}{N} \quad (3.20)$$

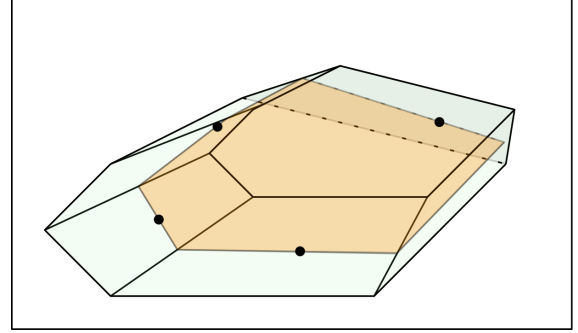
$$= \sup_{N \in \mathbb{N}} \frac{1 - \widetilde{\psi}_{T_i}(b - Np)}{N} \quad (3.21)$$

In the current example, $b = (-0.5, -0.5)$ and $p = (0.5, 0)$. Hence points of the form $b - Np$ correspond to a horizontal one-dimensional lattice. i.e., points of the form $-(N + 1)/2, -0.5$. Since all of these points are arbitrarily close to the side of $T_i + z$ for some $z \in \mathbb{Z}^2$ (as $i \rightarrow \infty$), $\widetilde{\psi}_{T_i}(b - Np) \geq 1 - \epsilon_i$ where $\epsilon_i \rightarrow 0$. This implies that the minimal lifting of the point could become arbitrarily close to zero, and the approximation $\frac{\widetilde{\psi}(p)}{\pi_{\min}(p)}$ could be arbitrarily poor.

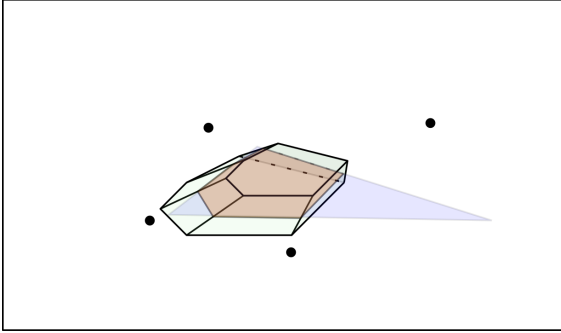
The proof for general $n \geq 2$ can be completed in two ways. One is a somewhat trivial way, by considering cylinders over the triangles considered above. A more involved construction considers the so-called *co-product* construction defined in Averkov and Basu [11], Basu and Paat [19], where one starts with the triangles defined above and iteratively takes a co-product with intervals to get maximal $b + \mathbb{Z}^n$ free sets in higher dimensions. It is not very hard to verify that the new sets continue to have minimal liftings which are arbitrarily better than the trivial lifting, because they contain a lower dimension copy of the triangle defined above. We do not provide more details, because this will involve definitions of the coproduct construction and other calculations which do not provide any additional insight, in our opinion.



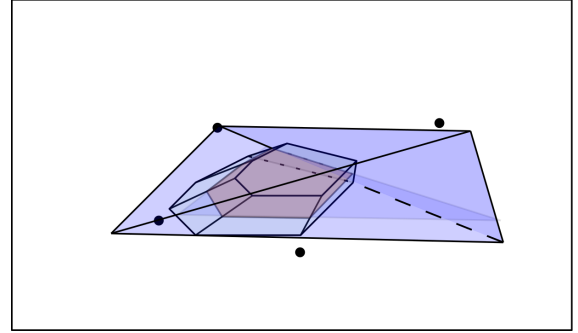
(a) A $b + \mathbb{Z}^n$ -free convex set that is to be approximated with a $b + \mathbb{Z}^n$ -free simplex.



(b) The integer lattice plane passing through the convex set is shown in orange.

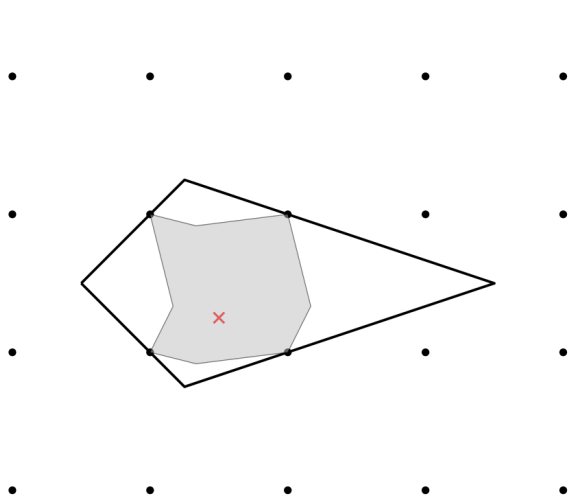


(c) The set shown in orange is a *lower-dimensional* $b + \mathbb{Z}^n$ -free convex set. This can be approximated by a lower-dimensional simplex using the induction hypothesis.

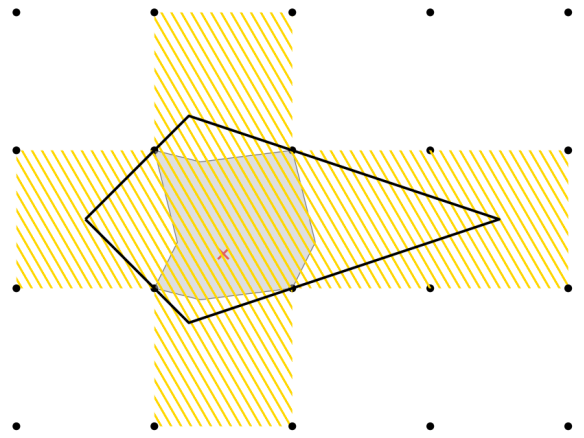


(d) Hyperplanes can be added that passes through the facets of the set in orange to get a truncated pyramid and then a simplex to approximate the given $b + \mathbb{Z}^n$ -free set.

Figure 3.3: Intuition behind [Lemma 48](#) to approximate a $b + \mathbb{Z}^n$ -free convex set with a simplex.



(a) Lifting region T of a generalized cross-polytope is shaded in grey. The red cross is the origin. Observe that the $T + \mathbb{Z}^n = \mathbb{R}^n$.



(b) Observe that the lifting region of a generalized cross-polytope is contained in n splits (shown in orange). This is the motivation behind [Algorithm 2](#).

Figure 3.4: The covering region of generalized cross-polytope and intuition behind [Algorithm 2](#)

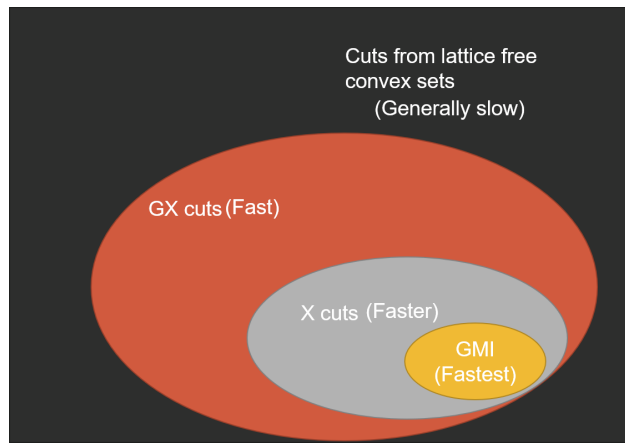


Figure 3.5: Venn diagram showing inclusions of various types of cuts and algorithmic efficiencies to generate them.

Algorithm 3 Computational testing procedure

Input: A mixed-integer program (MIP) in standard form. Number $N \geq 2$ of rows to use to

generate multi-row cuts; Number $k \geq 1$ of multi-row cuts; Number $\ell \geq 1$ of rounds of multi-row

cuts to be used; Number of $1 \leq q \leq N$ non-integer basics to be picked for GX-cuts.

1: $LP \leftarrow$ Objective of LP relaxation of MIP.

2: In the final simplex tableaux, apply GMI cuts on all rows whose corresponding basic variables are constrained to be integer in the original problem, but did not turn out to be integers.

3: $GMI \leftarrow$ Objective of LP relaxation of MIP and GMI cuts.

4: **for** i from 1 to ℓ **do**

5: **for** j from 1 to k **do**

6: Generate $\mu \in [0, 1]^N$ such that $\sum_{\xi=1}^N \mu_{\xi} = 1$. Also randomly select N rows where integrality constraints are violated for corresponding basic variables.

7: Generate an X-cut from the generated μ and the chosen set of rows.

8: Generate $f \in [0, 1]^N$ randomly.

9: Randomly select rows such that q of them correspond to rows that violate the integrality constraints and $N - q$ of them don't.

10: Generate a GX-cut from the generated μ , f and the set of rows.

11: **end for**

12: $(X_i, XG_i, GX_i, GXG_i) \leftarrow$ Objective of LP relaxation of MIP and all the (X-cuts, X-cuts and GMI, GX-cuts, GX-cuts and GMI).

13: **end for**

14: $X \leftarrow \max_{i=1}^{\ell} X_i$; $XG \leftarrow \max_{i=1}^{\ell} XG_i$; $GX \leftarrow \max_{i=1}^{\ell} GX_i$; $GXG \leftarrow \max_{i=1}^{\ell} GXG_i$.

15: $Best \leftarrow \max \{X, XG, GX, GXG\}$

16: **return** $LP, GMI, X, XG, GX, GXG, Best$

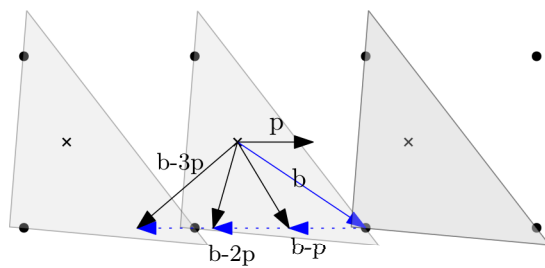


Figure 3.6: Example where trivial lifting can be very poor

Chapter 4

Complementarity problems

4.1 Introduction

In this chapter, we describe a well-studied branch of nonconvexity that arises from the presence of complementarity constraints. This corresponds to the constraint that two variables in the problem should remain non-negative while their product should be zero. Problems with this type of nonconvexity have a wide range of applications. [Theorem 18](#) shows how a game satisfying some regularity assumptions ([Assumptions A1 to A3](#)) can be posed as a complementarity problem. Abada et al. [\[1\]](#), Bushnell [\[36\]](#), Christensen and Siddiqui [\[41\]](#), Feijoo et al. [\[69\]](#), Gabriel et al. [\[75, 76\]](#), Huppmann and Egging [\[89\]](#), Martín et al. [\[112\]](#), Oke et al. [\[120\]](#), Zhuang and Gabriel [\[144\]](#) use complementarity problems to model markets from a game-theoretic perspective, where the complementarity conditions typically arise between the marginal profit and the quantity produced by the producer. In the field of mechanics, they typically arise in the context of frictional contact problems [\[102\]](#), where there is a complementarity relation between the frictional force between a pair of surfaces and the distance of separation between them. Wilmott et al. [\[139\]](#) show the

application of complementarity problems for pricing American options. A rigorous survey of their application is available in Ferris and Pang [71] and Facchinei and Pang [66, Chap 1.4, pg 20].

Before we delve into the new results we obtained as a part of this doctoral work, we define certain standard terms and results that we use in this section.

Facchinei and Pang [66] defines a general complementarity problem as follows.

Definition 50 (General complementarity problem). *Let $K \subseteq \mathbb{R}^n$ be a closed cone and $\langle \cdot, \cdot \rangle$ be an inner-product defined in \mathbb{R}^n . Let $F : K \mapsto \mathbb{R}^n$. Then $x \in \mathbb{R}^n$ solves the general complementarity problem if*

$$K \ni x \perp F(x) \in K^* \quad (4.1)$$

where K^* is the dual cone of K defined as $K^* := \{d \in \mathbb{R}^n : \langle v, d \rangle \geq 0, \forall v \in K\}$. The symbol $x \perp y$ refers to the requirement that $\langle x, y \rangle = 0$.

Remark 51. Note that if $K = \mathbb{R}_+^n$, the non-negative orthant, $\langle x, y \rangle := x^T y$ the standard inner-product, and $F(x) := Mx + q$, some affine function, the above problem is a linear complementarity problem as in Definition 11.

We formalize such a structure of K into the assumption below.

Assumption A4. *The cone K for the general complementarity problem is*

$$K = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ if } i \in \mathcal{I}\} \quad (4.2)$$

for some $\mathcal{I} \subseteq [n]$.

The special choice of K (which is a cartesian product of a non-negative orthant and a Euclidean space) as stated in Assumption A4, allows us to devise efficient algorithms to solve such problems. Also the dual cone K^* can be written in a simple closed form in such cases. We state that as a lemma below.

Lemma 52. *Let K satisfy [Assumption A4](#). The dual cone K^* is*

$$K^* = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} x_i \geq 0 & \text{if } i \in \mathcal{I} \\ x_i = 0 & \text{if } i \notin \mathcal{I} \end{array} \right\} \quad (4.3)$$

The structure that the above lemma guarantees, helps us design efficient algorithms for this problems. For conciseness, we will call the components of x corresponding to $i \in \mathcal{I}$ as positive variables (since they are constrained to be non-negative) and the components of x corresponding to $i \in [n] \setminus \mathcal{I}$ as free variables.

Solving complementarity problems We now briefly outline two methods to solve complementarity problems. The first is the motivation for the PATH algorithm [\[58\]](#) for solving complementarity problems. The following theorem is needed to motivate the PATH algorithm.

Theorem 53 ([\[66\]](#)). *Let $K \subseteq \mathbb{R}^n$ be a cone and $F : \mathbb{R}^n \mapsto \mathbb{R}^n$. Then x^* solves the general complementarity problem defined by K and F if and only if there exists $z^* \in \mathbb{R}^n$ such that $x^* = \text{Proj}_K(z^*)$ and $F_K^{\text{nor}}(z^*) = 0$, where $F_K^{\text{nor}} : K \mapsto \mathbb{R}^n$ is the normal map given F , defined by $F_K^{\text{nor}}(x) = F(\text{Proj}_K(x)) + x - \text{Proj}_K(x)$.*

This theorem allows us to rewrite the nonlinear complementarity problem as a zero-finding problem of F_K^{nor} which can be solved using Newton-like methods. However, the problem suffers from the fact that F_K^{nor} is not continuously differentiable since $\text{Proj}(\cdot)$ used to define it is not a continuously differentiable function. PATH solves this problem using a gradient smoothing-based method and a so-called stabilization scheme that they construct. However this comes at the cost that PATH converges in Q -superlinear rate instead of a quadratic rate associated with the Newton method [\[58, Theorem 8\]](#). Also under some assumptions, the algorithm is also guaranteed global convergence, unlike the Newton method, making it a popular choice to solve complementarity

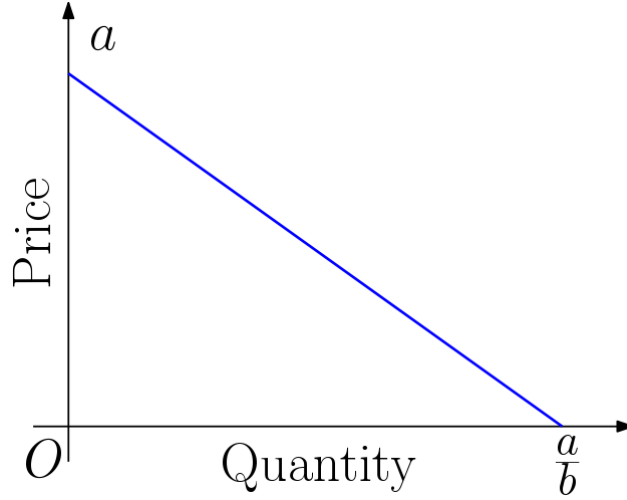


Figure 4.1: Demand curve showing the relation between price and quantity. Notice that lesser quantity is demanded at a higher price and vice versa.

problems. We refer more curious readers to Dirkse and Ferris [58] for a comprehensive description of this algorithm.

4.1.1 Example application

In this section, we formulate an example application of complementarity problems and motivate the economic significance of the complementarity constraint.

Nash-Cournot game - Oligopoly: Consider an energy market. Let the demand curve, the relation which connects the price vs quantity available, be defined by $P = a - bQ$, where P is the price of energy and Q is the total quantity of energy available in the market (Figure 4.1).

Consider two players A and B competitively producing energy non-cooperatively. Let us say both the players are interested in maximizing their profits. Further let the unit cost of production for A and B be c_A and c_B respectively.

Then the problem of A can be written as

$$\begin{aligned} \min_{q_A} \quad & : \quad c_A q_A - (a - b(q_A + q_B)) q_A && \text{subject to} \\ & q_A \geq 0 \end{aligned}$$

and the problem of B is

$$\begin{aligned} \min_{q_B} \quad & : \quad c_B q_B - (a - b(q_A + q_B)) q_B && \text{subject to} \\ & q_B \geq 0 \end{aligned}$$

The KKT conditions for these problems can be succinctly written as

$$0 \leq q_A \perp c_A - (a - b(q_A + q_B)) + b q_A \geq 0 \quad (4.4a)$$

$$0 \leq q_B \perp c_B - (a - b(q_A + q_B)) + b q_B \geq 0 \quad (4.4b)$$

We refer the readers to [Figure 2.1](#) for an intuition behind the KKT conditions. We observe that the problem defined by [eq. \(4.4\)](#) define the game played between A and B . The value q_A and q_B which solves [eq. \(4.4\)](#) is the Nash equilibrium to the game played by A and B .

Now with the preliminary definitions, properties and solution techniques of complementarity problems, we present our contribution in the following sections.

4.2 North American natural gas model

4.2.1 Introduction

With a wide range of application, understanding the behavior of complementarity problems gains importance, especially in cases where there are uncertainties in the problem. The behavior of a solution to a complementarity problem with random parameters was first addressed by Gürkan et al.

[86], where such problems were referred to as stochastic complementarity problems (SCP). Shanbhag [128] formally defines two primary formulations of SCPs, namely the almost-sure formulation and the expectation based formulation. While the former formulation rarely has a solution, the latter is widely accepted as the SCP. In addition to this, Chen and Fukushima [40] define an expected residual minimization (ERM) formulation for an SCP which they solve using Quasi-Monte Carlo methods. There have been a number of different methods employed to solve SCPs which include scenario-based methods, gradient-based methods, and Monte-Carlo sampling methods. Gabriel et al. [76] and Egging et al. [60] use a scenario reduction based approach to solve the SCP, which systematically analyzes the probability of a discrete set of scenarios. Not restricting to discrete distributions for the random variables, Jiang and Xu [95] provide an iterative line-search based algorithm to converge to a solution of the SCP under assumptions of monotonicity.

While the scenario reduction methods typically assume discrete probability distributions for random parameters, it is more appropriate to sample certain real-life quantities from a continuous distribution. For example, the Central Limit Theorem and the Extreme Value Theorem guarantee that processes arising as a sum of sufficiently many random variables and as maxima and minima of sufficiently many variables follow Normal or one of the extreme value distributions respectively [8, 78]. The ERM formulation and the solution by Chen and Fukushima [40] solve the problem of finding the mean of the solution irrespective of the distribution of the parameters. While Lamm et al. [103] compute confidence intervals for the solution of the expected value formulation of the problem, we do not have efficient methods to find the second-order statistics for large-scale complementarity problems, i.e., the covariance of the solution for problems over 10,000 output variables and over 1000 random input parameters.

Large-scale problems arise naturally out of detailed market models and there is considerable

interest in studying, understanding and solving such models. Gabriel et al. [74] discuss a case of an energy model with a large number of variables and parameters. Naturally, developing methods to solve such large-scale problems gained interest. Biller and Corlu [27], Contreras et al. [44], Luo et al. [111] discuss various tools ranging from mathematical techniques (Bender’s decomposition) to computational techniques (parallel processing) for solving large-scale optimization problems.

The objective of this work is to efficiently obtain second-order statistical information about solution vectors of large-scale stochastic complementarity problems. In addition, we also introduce a sensitivity metric which quantifies the change in uncertainty in the output due to a perturbation in the variance of uncertain parameters.

The diagonal elements of the covariance matrix, i.e., the variance of the solution vector, quantify the uncertainty in the decision variables while the off-diagonal elements capture the linear dependency among pairs of decision variables. Hyett et al. [90] and Benedetti-Cecchi [23] provide examples in the area of clinical pathways and ecology respectively about the utility of understanding the variance of the solution in addition to the mean. They also show that a knowledge of variance aids better understanding and planning of the system. Agrawal et al. [5] emphasize the necessity to understand covariance as a whole rather than individual variances by quantifying “the loss incurred on ignoring correlations” in a stochastic programming model.

The sensitivity metric developed in this work quantifies the sensitivity of the output uncertainty and thus helps us to directly compare input parameters by the amount of uncertainty they propagate to the solution.

In attaining the above objectives, apart from solving the stochastic complementarity problem in its expected value formulation, the most computationally expensive step is to solve a system of linear equations. We choose approximation methods over analytical methods, integration or Monte

Carlo simulation because of the computational hurdle involved while implementing those methods for large-scale problems. The method we describe in this section achieves the following:

- The most expensive step has to be performed just once, irrespective of the covariance of the input parameters. Once the linear system of equations is solved, for each given covariance scenario, we only perform two matrix multiplications.
- Approximating the covariance matrix and getting a sensitivity metric can be done by solving the said linear system just once.

The methods developed in this section can also be used for general differentiable optimization problems with linear equality constraints. We prove stronger results on error bounds for special cases of quadratic programming.

Having developed this method, we apply it to a large-scale stochastic natural gas model for North America, an extension of the deterministic model developed by Feijoo et al. [69] and determine the covariance of the solution variables. We then proceed to identify the parameters which have the greatest impact on the solution. A Python class for efficiently storing and operating on sparse arrays of dimension greater than two is created. This is useful for working with high-dimensional problems which have an inherent sparse structure in the gradients.

We divide this as follows. [Section 4.2.2](#) formulates the problem and mentions the assumptions used in the work. It then discusses the method used to solve the stochastic complementarity problem, develops the algorithm used to approximate the solution covariance and provides proofs for bounding the error. [Section 4.2.3](#) develops a framework to quantify the sensitivity of the solution to each of the random variables. [Section 4.2.4](#) shows how the result can be applied to certain optimization problems with equality constraints. Having obtained the theoretical results, [Section 4.2.5](#) gives an example of an oligopoly where this method can be applied. [Section 4.2.6](#)

describes the Natural Gas Model to which the said method is applied. It also discusses the scenario we used on the model and the results hence obtained. [Section 4.2.7](#) discusses the possible enhancements for the model and its limitations in the current form.

4.2.2 Approximation of covariance

The approximation algorithm

Definition 54. *Given $\mathbf{F} : \mathbb{R}^{n \times m} \mapsto \mathbb{R}^n$, and parameters $\theta \in \mathbb{R}^m$, the parametrized complementarity problem is to find $\mathbf{x} \in \mathbb{R}^n$ such that*

$$\mathbb{K} \ni \mathbf{x} \perp \mathbf{F}(\mathbf{x}; \theta) \in \mathbb{K}^* \quad (4.5)$$

We now define C-functions, which are central to pose the complementarity problem into an unconstrained optimization problem. The equivalent formulation as an unconstrained optimization problem assists us in developing the algorithm.

Definition 55. *[66] A function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a C-function when*

$$\left. \begin{aligned} \psi(x, y) &= 0 \\ &\Leftrightarrow \\ x \geq 0 \quad y \geq 0 \quad xy &= 0 \end{aligned} \right\} \quad (4.6)$$

We consider the following commonly used C-functions.

$$\psi_{FB}(x, y) = \sqrt{x^2 + y^2} - x - y \quad (4.7)$$

$$\psi_{min}(x, y) = \min(x, y) \quad (4.8)$$

$$\psi_{Man}(x, y) = \zeta(|x - y|) - \zeta(|x|) - \zeta(|y|) \quad (4.9)$$

where $\zeta(x)$ is some strictly monotonic real-valued function with $\zeta(0) = 0$

Under our assumptions on \mathbb{K} , the following two theorems show the equivalence of the complementarity problem and an unconstrained minimization problem. The first theorem shows the necessary condition for a solution of the complementarity problem.

Theorem 56. *Suppose [Assumption A4](#) holds. Then every solution $\mathbf{x}^*(\theta)$ of the parameterized complementarity problem in [\(4.5\)](#), is a global minimum of the following function $f(\mathbf{x}; \theta)$,*

$$\Phi_i(\mathbf{x}, \theta; \mathbf{F}) = \begin{cases} \mathbf{F}_i(\mathbf{x}, \theta) & \text{if } i \notin \mathcal{I} \\ \psi_i(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}, \theta)) & \text{if } i \in \mathcal{I} \end{cases} \quad (4.10)$$

$$f(\mathbf{x}; \theta) = \frac{1}{2} \|\Phi(\mathbf{x}; \theta; \mathbf{F})\|_2^2 \quad (4.11)$$

with an objective value 0, for some set of not necessarily identical C-functions ψ_i .

Proof. Since \mathbf{x}^* solves the problem, following from the requirement that $\mathbf{F}(\mathbf{x}^*) \in \mathbb{K}^*$ and [Lemma 52](#), if $i \notin \mathcal{I}$, $\mathbf{F}_i(\mathbf{x}^*(\theta); \theta) = 0$.

For $i \in \mathcal{I}$, $\mathbf{x}^* \in \mathbb{K} \Rightarrow \mathbf{x}^*_i \geq 0$ and $\mathbf{F}(\mathbf{x}^*) \in \mathbb{K}^* \Rightarrow \mathbf{F}_i(\mathbf{x}^*) \geq 0$. Also from the requirement $\mathbf{x}^{*T} \mathbf{F}(\mathbf{x}^*) = 0$, one of the above two quantities should vanish for each $i \in \mathcal{I}$. But C-functions are precisely functions that vanish when both their arguments are non-negative and of them equal zero. So $\psi_i(\mathbf{x}^*, \mathbf{F}_i(\mathbf{x}^*)) = 0$.

Thus each coordinate of Φ is individually zero, which makes $f(\mathbf{x}^*)$ vanish, which is the smallest value f can take. Thus \mathbf{x}^* is a global minimum of f . ■

Now we show that the sufficient condition for a point to be a solution of the complementarity problem is that it should be a global minimum to the same unconstrained optimization problem as in [Theorem 56](#).

Theorem 57. *Suppose [Assumption A4](#) holds. If a solution to the problem in [\(4.5\)](#) exists and $\mathbf{x}^*(\theta)$ is an unconstrained global minimizer of $f(\mathbf{x}; \theta)$ defined in [\(4.11\)](#), then $\mathbf{x}^*(\theta)$ solves the*

complementarity problem in (4.5).

Proof. Since a solution exists for the NCP, we know by Theorem 56 that the minimum value \mathbf{f} can take is 0. Suppose we have $\mathbf{x}^* \in \mathbb{R}^n$ such that $\mathbf{f}(\mathbf{x}^*; \theta) = 0$. Since \mathbf{f} is sum of squares, this can happen only if each of the individual terms are zero. This means for $i \notin \mathcal{I}$, $\mathbf{F}_i(\mathbf{x}^*) = 0$.

Now since $\psi_i(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}^*)) = 0$ for $i \in \mathcal{I}$, we know $\mathbf{F}_i(\mathbf{x}^*) \geq 0$. This combined with the previous point implies $\mathbf{F}(\mathbf{x}^*) \in \mathbb{K}^*$.

Also from the fact that $\psi_i(\mathbf{x}_i^*; \mathbf{F}_i(\mathbf{x}^*; \theta)) = 0$ for $i \in \mathcal{I}$, we know that $\mathbf{x}^* \in \mathbb{K}$. It also implies that $\mathbf{x}_i^* \mathbf{F}_i(\mathbf{x}^*) = 0$ for $i \in \mathcal{I}$. Thus

$$\mathbf{x}^{*T} \mathbf{F}(\mathbf{x}^*; \theta) = \sum_{i=1}^n \mathbf{x}_i^* \mathbf{F}_i \quad (4.12)$$

$$= \sum_{i \in \mathcal{I}} \mathbf{x}_i^* \mathbf{F}_i + \sum_{i \notin \mathcal{I}} \mathbf{x}_i^* \mathbf{F}_i \quad (4.13)$$

$$= 0 + 0 = 0 \quad (4.14)$$

This implies $\mathbf{x}^*(\theta) \perp \mathbf{F}(\mathbf{x}^*; \theta)$ and $\mathbf{x}^*(\theta)$ solves the complementarity problem. ■

We state the below corollary explicitly to show that the equivalence holds for a stochastic complementarity problem too.

Corollary 58. *A solution \mathbf{x}^* of a stochastic complementarity problem with $\widehat{\mathbf{F}} \equiv \mathbb{E}\mathbf{F}(\mathbf{x}; \theta(\omega))$ is a global minimum of the function \mathbf{f} defined in (4.11).*

Proof. The proof is immediate by considering a new function $\widetilde{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that $\widetilde{F}(\mathbf{x}) = \mathbb{E}\mathbf{F}(\mathbf{x}; \theta(\omega))$. ■

Now given a function \mathbf{F} , and a set \mathbb{K} which satisfies Assumption A4, and a solution of the NCP $\mathbf{x}^*(\mathbb{E}(\theta))$ for some fixed $\theta = \mathbb{E}(\theta)$, we define a vector valued function $\Phi : \mathbb{R}^{n \times m} \mapsto \mathbb{R}^n$ component-

wise as follows.

$$\Phi_i(\mathbf{x}, \theta; \mathbf{F}) = \begin{cases} \mathbf{F}_i(\mathbf{x}, \theta) & \text{if } i \notin \mathcal{I} \\ \psi^2(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}, \theta)) & \text{if } i \in \mathcal{Z} \\ \psi(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}, \theta)) & \text{otherwise} \end{cases} \quad (4.15)$$

where

$$\mathcal{Z} = \{i \in \mathbb{I} : \mathbf{x}_i^*(\mathbb{E}(\theta)) = \mathbf{F}_i(\mathbf{x}^*(\mathbb{E}(\theta)); \mathbb{E}(\theta)) = 0\} \quad (4.16)$$

$$\mathbf{f}(\mathbf{x}; \theta) = \frac{1}{2} \|\Phi(\mathbf{x}; \mathbf{F})\|_2^2 \quad (4.17)$$

Note that if ψ is a C-function, ψ^2 is also a C-function since $\psi^2 = 0 \iff \psi = 0$. We observe from [Theorems 56](#) and [57](#) that minimizing $\mathbf{f}(\mathbf{x}; \theta)$ over \mathbf{x} is equivalent to solving the NCP in [\(4.5\)](#).

Now we assume conditions on the smoothness of \mathbf{F} so that the solution to a perturbed problem is sufficiently close to the original solution.

Assumption A5. $\mathbf{F}(\mathbf{x}; \theta)$ is twice continuously differentiable in \mathbf{x} and θ over an open set containing \mathbb{K} and $\mathbb{E}(\theta)$.

Given that the rest of the analysis is on the function $\mathbf{f}(\mathbf{x}; \theta)$ defined in [\(4.11\)](#), we prove that a sufficiently smooth \mathbf{F} and a suitable ψ ensure a sufficiently smooth \mathbf{f} .

Theorem 59. With [Assumption A5](#) holding, we state $\mathbf{f}(\mathbf{x}; \mathbb{E}(\theta))$ defined as in [\(4.17\)](#) is a twice continuously differentiable at $\mathbf{f}(\mathbf{x}; \mathbb{E}(\theta)) = 0$ for any C function ψ provided

1. ψ is twice differentiable at $\{(a, b) \in \mathbb{R}^2 : \psi(a, b) = 0\} \setminus \{(0, 0)\}$ with a finite derivative and finite second derivative.
2. ψ vanishes sufficiently fast near the origin. i.e.,

$$\lim_{(a,b) \rightarrow (0,0)} \psi^2(a, b) \frac{\partial^2 \psi(a, b)}{\partial a \partial b} = 0 \quad (4.18)$$

Proof. Given that \mathbf{f} is a sum of squares, it is sufficient to prove each term individually is twice continuously differentiable to prove the theorem. Also since we are only interested where \mathbf{f} vanishes, it is sufficient to prove the above property for each term where it vanishes.

Consider terms from $i \notin \mathcal{I}$. Since \mathbf{F}_i is twice continuously differentiable, \mathbf{F}_i^2 is twice continuously differentiable too.

Consider the case $i \in \mathcal{I}$; $\mathbf{x}_i^*(\mathbb{E}(\theta)) = \mathbf{F}_i(\mathbf{x}_i^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) = 0$. These contribute a ψ^4 term to \mathbf{f} .

With the notation $\psi_i = \psi(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}; \theta))$, we clearly have,

$$\frac{\partial \psi_i^4}{\partial \mathbf{x}_j} = 4\psi_i^3 \left(\frac{\partial \psi_i}{\partial a} \delta_{ij} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} \right) = 0 \quad (4.19)$$

$$\begin{aligned} \frac{\partial^2 \psi_i^4}{\partial \mathbf{x}_j \partial \mathbf{x}_k} &= 12\psi_i^2 \left(\frac{\partial \psi_i}{\partial a} \delta_{ij} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} \right) \left(\frac{\partial \psi_i}{\partial a} \delta_{ik} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_k} \right) \\ &\quad + 4\psi_i^3 \left(\frac{\partial^2 \psi_i}{\partial a^2} \delta_{ij} \delta_{ik} + \frac{\partial^2 \psi_i}{\partial a \partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_k} + \text{other terms} \right) \end{aligned} \quad (4.20)$$

$$= 0 \quad (4.21)$$

For the third case, we have

$$\frac{\partial \psi_i^2}{\partial \mathbf{x}_j} = 2\psi_i \left(\frac{\partial \psi_i}{\partial a} \delta_{ij} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} \right) = 0 \quad (4.22)$$

$$\begin{aligned} \frac{\partial^2 \psi_i^2}{\partial \mathbf{x}_j \partial \mathbf{x}_k} &= \left(\frac{\partial \psi_i}{\partial a} \delta_{ij} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} \right) \\ &\quad + 2\psi_i \left(\frac{\partial^2 \psi_i}{\partial a^2} \delta_{ij} \delta_{ik} + \frac{\partial^2 \psi_i}{\partial a \partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_k} + \text{other terms} \right) \end{aligned} \quad (4.23)$$

$$= \frac{\partial \psi_i}{\partial a} \delta_{ij} + \frac{\partial \psi_i}{\partial b} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} \quad (4.24)$$

Continuity of \mathbf{f} at the points of interest follow the continuity of the individual terms at the points. ■

The following corollaries show the existence of C-functions ψ which satisfy the hypothesis of [Theorem 59](#).

Corollary 60. *With [Assumption A5](#) holding, for the choice of C-function $\psi = \psi_{FB}$ defined in [\(4.7\)](#), the function \mathbf{f} is twice continuously differentiable.*

Proof. For $\psi = \psi_{FB}$, we have assumption 1 of [Theorem 59](#) satisfied by [\[66\]](#). For assumption 2,

$$\lim_{(a,b) \rightarrow (0,0)} \psi^2(a,b) \frac{\partial^2 \psi(a,b)}{\partial a \partial b} = \lim_{(a,b) \rightarrow (0,0)} \left(\sqrt{a^2 + b^2} - a - b \right)^2 \frac{ab}{\left(\sqrt{a^2 + b^2} \right)^3} \quad (4.25)$$

$$= 0 \quad (4.26)$$

Thus \mathbf{f} is twice continuously differentiable at its zeros. The twice continuous differentiability elsewhere follows directly from the fact that ψ_{FB} is twice continuously differentiable everywhere except at the origin. This ensures that all the terms in the derivative of the sum of squares exist and are finite. ■

We now define an isolated solution to a problem and assume that the problem of interest has this property. This is required to ensure that our approximation is well defined.

Definition 61. [\[119\]](#) *A solution \mathbf{x}^* of a problem is said to be an isolated solution, if there is a neighborhood $\mathcal{B}(\mathbf{x}^*; \epsilon)$ of \mathbf{x}^* , where \mathbf{x}^* is the only solution of the problem.*

A counter-example for an isolated minimum is shown on the left of [Figure 4.2](#). It is a plot of the function

$$f(x) = 5x^2 + x^2 \sin\left(\frac{1}{x^2}\right) \quad (4.27)$$

and the global minimum at $x = 0$ is not an isolated minimum as we can confirm that any open interval around $x = 0$ has other minimum contained in it. Unlike this case, in this chapter, we assume that if we obtain a global-minimum of \mathbf{f} , then it is an isolated minimum.

Assumption A6. *For an ε ball around $\theta = \mathbb{E}(\theta)$, there exists a known solution $\mathbf{x}^*(\mathbb{E}(\theta))$ such that it is an isolated first-order stationary point of $\mathbf{f}(\mathbf{x}; \theta)$.*

Algorithm 4 Approximating Covariance

Solve the complementarity problem in (4.5) for the mean value of $\theta = \mathbb{E}(\theta)$ and calibrate the value of the parameters $\theta = \mathbb{E}(\theta)$ for this solution. Call this solution as \mathbf{x}^* . Choose a tolerance level τ .

- 1: Evaluate $\mathbf{F}^* \leftarrow \mathbf{F}(\mathbf{x}^*; \mathbb{E}(\theta))$, $G_{ij} \leftarrow \frac{\partial \mathbf{F}_i(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \mathbf{x}_j}$, $L_{ij} \leftarrow \frac{\partial \mathbf{F}_i(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \theta_j}$.
- 2: Choose a C-function ψ such that the conditions in Theorem 59 are satisfied.
- 3: Define the function $\psi^a(a, b) = \frac{\partial \psi(a, b)}{\partial a}$, $\psi^b(a, b) = \frac{\partial \psi(a, b)}{\partial b}$.
- 4: Find the set of indices $\mathcal{Z} = \{z \in \mathcal{I} : |\mathbf{x}_z^*| = |\mathbf{F}_z^*| \leq \tau\}$.
- 5: Define

$$\mathcal{M}_{ij} \leftarrow \begin{cases} G_{ij} & \text{if } i \notin \mathcal{I} \\ 0 & \text{if } i \in \mathcal{Z} \\ \psi^a(\mathbf{x}_i^*, \mathbf{F}_i^*)\delta_{ij} + \psi^b(\mathbf{x}_i^*, \mathbf{F}_i^*)G_{ij} & \text{otherwise} \end{cases} \quad (4.28)$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- 6: Define

$$\mathcal{N}_{ij} \leftarrow \begin{cases} L_{ij} & \text{if } i \notin \mathcal{I} \\ 0 & \text{if } i \in \mathcal{Z} \\ \psi^b(\mathbf{x}_i^*, \mathbf{F}_i^*)L_{ij} & \text{otherwise} \end{cases} \quad (4.29)$$

- 7: Solve the linear systems of equations for \mathcal{T} .

$$\mathcal{M}\mathcal{T} = \mathcal{N} \quad (4.30)$$

If \mathcal{M} is non singular, we have a unique solution. If not, a least square solution or a solution obtained by calculating the Moore Penrose Pseudo inverse [87] can be used.

- 8: Given \mathcal{C} , a covariance matrix of the input random parameters, $\theta(\omega)$, **return** $\mathcal{C}^* \leftarrow \mathcal{T}\mathcal{C}\mathcal{T}^T$.
-

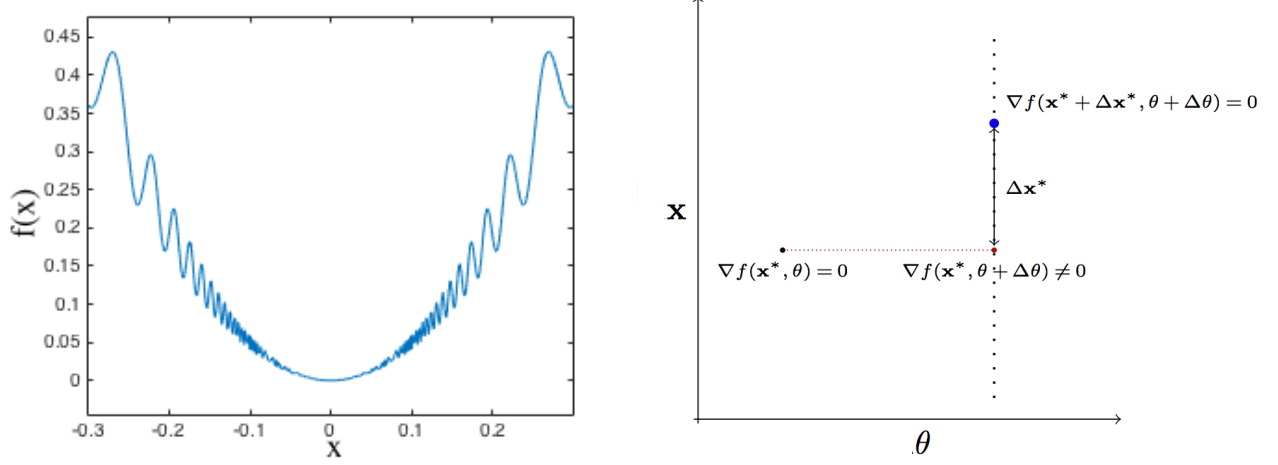


Figure 4.2: Left: An example of a function where the global minimum $x = 0$ is a non-isolated solution. Right: The intuition behind our approximation for finding where $\nabla f(\mathbf{x}, \theta) = 0$ under a small perturbation.

Theorem 62 ([125]). *Algorithm 4 generates a first-order approximation for the change in solution for a perturbation in parameters and computes the covariance of the solution for a complementarity problem with uncertain parameters with small variances.*

Proof. Consider the function $f(\mathbf{x}; \theta)$. From Theorem 56, \mathbf{x}^* minimizes this function for $\theta = \mathbb{E}(\theta)$. From Theorem 59, we have $f(\mathbf{x}; \theta)$ is twice continuously differentiable at all its zeros. Thus we have,

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*; \mathbb{E}(\theta)) = 0 \quad (4.31)$$

Now suppose the parameters $\mathbb{E}(\theta)$ are perturbed by $\Delta\theta$, then the above gradient can be written using the mean value theorem and then approximated up to the first order as follows.

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta) + \Delta\theta) = \nabla_{\mathbf{x}} f(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) + \nabla_{(\theta)} \nabla_{\mathbf{x}} f(\mathbf{x}^*(\mathbb{E}(\theta)), \tilde{\theta}) \Delta\theta \quad (4.32)$$

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta) + \Delta\theta) - \nabla_{\mathbf{x}} f(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) \approx \mathcal{J} \Delta\theta \quad (4.33)$$

where,

$$\tilde{\theta} \in [\theta, \theta + \Delta\theta] \quad (4.34)$$

$$\mathcal{J}_{ij} = [\nabla_{(\theta)} \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta))]_{ij} \quad (4.35)$$

$$= \frac{\partial [\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*; \mathbb{E}(\theta))]_i}{\partial (\theta)_j} \quad (4.36)$$

Since $\mathcal{J}\Delta\theta$ is not guaranteed to be 0, we might have to alter \mathbf{x} to bring the gradient back to zero.

i.e., we need $\Delta\mathbf{x}$ such that $\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)) + \Delta\mathbf{x}, \mathbb{E}(\theta) + \Delta\theta) = \mathbf{0}$. But by the mean value theorem,

$$\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)) + \Delta\mathbf{x}, \mathbb{E}(\theta) + \Delta\theta) = \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta) + \Delta\theta) + \nabla_{\mathbf{x}}^2 \mathbf{f}(\tilde{\mathbf{x}}, \mathbb{E}(\theta) + \Delta\theta) \Delta\mathbf{x} \quad (4.37)$$

$$\mathbf{0} \approx \mathcal{J}\Delta\theta + \nabla_{\mathbf{x}}^2 \mathbf{f}(\tilde{\mathbf{x}}, \mathbb{E}(\theta)) \Delta\mathbf{x} \quad (4.38)$$

$$\approx \mathcal{J}\Delta\theta + \nabla_{\mathbf{x}}^2 \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) \Delta\mathbf{x} \quad (4.39)$$

$$\mathcal{H}\Delta\mathbf{x} \approx -\mathcal{J}\Delta\theta \quad (4.40)$$

where,

$$\tilde{\mathbf{x}} \in [\mathbf{x}^*(\mathbb{E}(\theta)), \mathbf{x}^*(\mathbb{E}(\theta)) + \Delta\mathbf{x}] \quad (4.41)$$

$$[\mathcal{H}]_{ij} = [\nabla_{\mathbf{x}}^2 \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta))]_{ij} \quad (4.42)$$

$$= \frac{\partial [\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*; \mathbb{E}(\theta))]_i}{\partial \mathbf{x}_j} \quad (4.43)$$

Now from [119], the gradient of the least squares function \mathbf{f} can be written as

$$\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*, \mathbb{E}(\theta)) = \mathcal{M}^T \Phi(\mathbf{x}^*, \mathbb{E}(\theta)) \quad (4.44)$$

$$[\mathcal{M}]_{ij} = \frac{\partial \Phi_i(\mathbf{x}^*, \mathbb{E}(\theta))}{\partial \mathbf{x}_j} \quad (4.45)$$

$$= \begin{cases} \frac{\partial \mathbf{F}_i(\mathbf{x}^*, \mathbb{E}(\theta))}{\partial \mathbf{x}_j} & \text{if } i \notin \mathcal{I} \\ \frac{\partial \psi^2(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}^*, \mathbb{E}(\theta)))}{\partial \mathbf{x}_j} & \text{if } i \in \mathcal{Z} \\ \frac{\partial \psi(\mathbf{x}_i, \mathbf{F}_i(\mathbf{x}^*, \mathbb{E}(\theta)))}{\partial \mathbf{x}_j} & \text{otherwise} \end{cases} \quad (4.46)$$

$$= \begin{cases} \frac{\partial \mathbf{F}_i(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \mathbf{x}_j} & \text{if } i \notin \mathcal{I} \\ 0 & \text{if } i \in \mathcal{Z} \\ \frac{\partial \psi_i}{\partial \mathbf{x}_j} & \text{otherwise} \end{cases} \quad (4.47)$$

which is the form of \mathcal{M} defined in [Algorithm 4](#). Also

$$\mathcal{H} = \nabla_{\mathbf{x}}^2 f(\mathbf{x}^*; \mathbb{E}(\theta)) = \mathcal{M}^T \mathcal{M} + \sum_{i=1}^n \Phi_i(\mathbf{x}^*; \mathbb{E}(\theta)) \nabla_{\mathbf{x}}^2 \Phi_i(\mathbf{x}^*; \mathbb{E}(\theta)) \quad (4.48)$$

$$= \mathcal{M}^T \mathcal{M} \quad (4.49)$$

where the second term vanishes since we have from [Theorem 56](#) that each term of Φ individually vanishes at the solution. Now

$$\mathcal{J} = \nabla_{\mathbf{x}\theta} f(\mathbf{x}^*; \mathbb{E}(\theta)) \quad (4.50)$$

$$\mathcal{J}_{ij} = \frac{\partial [\nabla_{\mathbf{x}} f(\mathbf{x}^*; \mathbb{E}(\theta))]_i}{\partial \theta_j} \quad (4.51)$$

$$= \frac{\partial}{\partial \theta_j} \left(\sum_{k=1}^n [\nabla_{\mathbf{x}} \Phi(\mathbf{x}^*; \mathbb{E}(\theta))]_{ki} \Phi_k(\mathbf{x}^*; \mathbb{E}(\theta)) \right) \quad (4.52)$$

$$= \sum_{k=1}^n \left(\frac{\partial [\nabla_{\mathbf{x}} \Phi(\mathbf{x}^*; \mathbb{E}(\theta))]_{ki}}{\partial \theta_j} \Phi_k(\mathbf{x}^*; \mathbb{E}(\theta)) + [\nabla_{\mathbf{x}} \Phi(\mathbf{x}^*; \mathbb{E}(\theta))]_{ki} \frac{\partial \Phi_k(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \theta_j} \right) \quad (4.53)$$

$$= \sum_{k=1}^n \mathcal{M}_{ki} \mathcal{N}_{kj} = \mathcal{M}^T \mathcal{N} \quad (4.54)$$

where the first term vanished because Φ_i are individually zeros, and we define

$$\mathcal{N}_{ij} = \frac{\partial \Phi_k(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \theta_j} \quad (4.55)$$

$$= \begin{cases} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} & \text{if } i \notin \mathcal{I} \\ 2\psi(\mathbf{x}^*_i; \mathbf{F}_i^*) \psi^b(\mathbf{x}^*_i; \mathbf{F}_i^*) \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} & \text{if } i \in \mathcal{Z} \\ \psi^b(\mathbf{x}^*_i; \mathbf{F}_i^*) \frac{\partial \mathbf{F}_i(\mathbf{x}^*; \mathbb{E}(\theta))}{\partial \mathbf{x}_j} & \text{otherwise} \end{cases} \quad (4.56)$$

$$= \begin{cases} \frac{\partial \mathbf{F}_i}{\partial \mathbf{x}_j} & \text{if } i \notin \mathcal{I} \\ 0 & \text{if } i \in \mathcal{Z} \\ \psi^b(\mathbf{x}_i^*; \mathbf{F}_i^*) \frac{\partial \mathbf{F}_i(\mathbf{x}_i^*; \mathbb{E}(\theta))}{\partial \mathbf{x}_j} & \text{otherwise} \end{cases} \quad (4.57)$$

which is the form of \mathcal{N} defined in [Algorithm 4](#). By [Assumption A6](#), we have a unique minimum in the neighborhood of \mathbf{x}^* where the gradient vanishes. So we have from [\(4.40\)](#), [\(4.49\)](#) and [\(4.54\)](#)

$$\mathcal{H}\Delta\mathbf{x} = -\mathcal{J}\Delta\theta \quad (4.58)$$

$$\mathcal{M}^T \mathcal{M} \Delta\mathbf{x} = -\mathcal{M}^T \mathcal{N} \Delta\theta \quad (4.59)$$

$\Delta\mathbf{x}$ solves the above equation, if it solves

$$\mathcal{M} \Delta\mathbf{x} = -\mathcal{N} \Delta\theta \quad (4.60)$$

By defining \mathcal{T} as the solution to the linear system of equations

$$\mathcal{M} \mathcal{T} = \mathcal{N} \quad (4.61)$$

$$\Delta\mathbf{x} = -\mathcal{T} \Delta\theta \quad (4.62)$$

and we have the above first-order approximation. We know that if some vector \mathbf{x} has covariance \mathcal{C} , then for a matrix A , the vector $A\mathbf{x}$ will have covariance ACA^T . So we have.

$$\text{cov}(\Delta\mathbf{x}) \approx \mathcal{T} \text{cov}(\Delta\theta) \mathcal{T}^T \quad (4.63)$$

This forms the basis for the computation of the covariance of $\Delta\mathbf{x}$ in [Algorithm 4](#). ■

For computational purposes, the matrix \mathcal{T} in the above equation has to be calculated only once, irrespective of the number of scenarios for which we would like to run for the covariance of θ . Thus if $\mathbf{x} \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ and we want to test the output covariance for k different input covariance cases, the complexity is equal to that of solving a system of n linear equations m times as in [\(4.61\)](#), and

hence is $O(mn^2)$. i.e., the complexity is quadratic in the number of output variables, linear in the number of input parameters and constant in the number of covariance scenarios we would like to run.

In [Theorem 63](#) below, we prove that the error in the approximation of [Theorem 62](#) can be bounded using the condition number of the Hessian. We need the following assumption that the condition number of the Hessian of \mathbf{f} is bounded and the Hessian is Lipschitz continuous.

Assumption A7. *At the known solution of the complementarity problem of interest ($\theta = \mathbb{E}(\theta)$),*

1. *The condition number of the Hessian of \mathbf{f} defined is finite and equal to κ_H*
2. *The Hessian of \mathbf{f} is Lipschitz continuous with a Lipschitz constant $\mathcal{L}(\mathbf{x}^*; \theta)$.*

Theorem 63 ([125]). *With [Assumption A7](#) holding, the error in the linear [eq. \(4.40\)](#) for a perturbation of ϵ is $o(\epsilon)$.*

Proof. Since $\nabla^2 f$ is Lipschitz continuous on both \mathbf{x} and θ , we can write for $\tilde{\mathbf{x}}$ near \mathbf{x}^* ,

$$\left\| \nabla_{\mathbf{x}}^2 \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) - \nabla_{\mathbf{x}}^2 \mathbf{f}(\tilde{\mathbf{x}}, \mathbb{E}(\theta)) \right\| \leq \mathcal{L}(\mathbf{x}^*; \theta) \|\mathbf{x}^*(\mathbb{E}(\theta)) - \tilde{\mathbf{x}}\| \quad (4.64)$$

$$\leq \mathcal{L}(\mathbf{x}^*; \theta) \|\Delta \mathbf{x}\| \quad (4.65)$$

$$\tilde{\mathcal{H}} = \nabla_{\mathbf{x}}^2 \mathbf{f}(\tilde{\mathbf{x}}, \mathbb{E}(\theta)) \quad (4.66)$$

$$= \mathcal{H} + \varepsilon_H \quad (4.67)$$

where $\|\varepsilon_H\| \leq \mathcal{L}(\mathbf{x}^*; \theta) \|\Delta \mathbf{x}\|$. Applying the Lipschitz continuity on θ ,

$$\left\| \nabla_{\theta} \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \tilde{\theta}) - \nabla_{\theta} \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \mathbb{E}(\theta)) \right\| \leq \mathcal{L}(\mathbf{x}^*; \theta) \|\tilde{\theta} - \mathbb{E}(\theta)\| \quad (4.68)$$

$$\leq \mathcal{L}(\mathbf{x}^*; \theta) \|\Delta \theta\| \quad (4.69)$$

$$\tilde{\mathcal{J}} = \nabla_{\theta} \nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x}^*(\mathbb{E}(\theta)), \tilde{\theta}) \quad (4.70)$$

$$= \mathcal{J} + \varepsilon_J \quad (4.71)$$

where $\|\varepsilon_J\| \leq \mathcal{L}(\mathbf{x}^*; \theta) \|\Delta\theta\|$. Thus the equation

$$\tilde{\mathcal{H}}\Delta\mathbf{x} = \tilde{\mathcal{J}}\Delta\theta \quad (4.72)$$

is exact, even if we cannot compute $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{J}}$ exactly. Now the error in inverting $\tilde{\mathcal{H}}$ is bounded by the condition number [87].

$$\frac{\|\mathcal{H}^{-1} - \tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}^{-1}\|} \leq \frac{\kappa_H \frac{\|\varepsilon_H\|}{\|\tilde{\mathcal{H}}\|}}{1 - \kappa_H \frac{\|\varepsilon_H\|}{\|\tilde{\mathcal{H}}\|}} \quad (4.73)$$

Assuming $\kappa_H \|\varepsilon_H\| \ll \|\mathcal{H}\|$, the above equation becomes

$$\frac{\|\mathcal{H}^{-1} - \tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}^{-1}\|} \leq \kappa_H \frac{\|\varepsilon_H\|}{\|\tilde{\mathcal{H}}\|} \quad (4.74)$$

$$\Rightarrow \|\tilde{\mathcal{H}}^{-1} - \mathcal{H}^{-1}\| \leq \kappa_H \frac{\|\tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}\|} \|\varepsilon_H\| \quad (4.75)$$

$$\Rightarrow \|\tilde{\mathcal{H}}^{-1}\tilde{\mathcal{J}} - \mathcal{H}^{-1}\tilde{\mathcal{J}} - \mathcal{H}^{-1}\varepsilon_J\| \leq \kappa_H \frac{\|\tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}\|} \varepsilon_H \|\mathcal{J}\| + \kappa_H \frac{\|\tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}\|} \|\varepsilon_H\| \|\varepsilon_J\| \quad (4.76)$$

$$\Rightarrow \|\tilde{\mathcal{H}}^{-1}\tilde{\mathcal{J}} - \mathcal{H}^{-1}\mathcal{J}\| \leq k_1 \|\Delta\mathbf{x}\| + k_2 \|\Delta\theta\| \quad (4.77)$$

with

$$k_1 = \kappa_H \mathcal{L}(\mathbf{x}^*; \theta) \frac{\|\tilde{\mathcal{H}}^{-1}\|}{\|\tilde{\mathcal{H}}\|} \|\mathcal{J}\| \quad (4.78)$$

$$k_2 = \mathcal{L}(\mathbf{x}^*; \theta) \|\mathcal{H}^{-1}\| \quad (4.79)$$

Thus we have from (4.77), that the error in the approximation done in Algorithm 4 is bounded. ■

4.2.3 Stochastic sensitivity analysis

In this section, we quantify the sensitivity of the solution to each of the input parameters. To achieve this, we define total linear sensitivity. We then show how these quantities can be calculated

using the matrix \mathcal{T} derived earlier. We then proceed to prove that these quantities also bound the maximum increase in uncertainties of the output. We now define a quantity that bounds a maximum change in the value of a function due to perturbations.

Definition 64. Given a function $\lambda : \mathbb{R}^m \mapsto \mathbb{R}^n$, the total linear sensitivity, $\beta_d \in \mathbb{R}_+$ of a dimension $d \leq m$; $d \in \mathbb{N}$ at a point $\mathbf{x} \in \mathbb{R}^m$ is defined for $\delta > 0$, sufficiently small,

$$\beta_d = \inf \{ \alpha : \|\lambda(\mathbf{x})\|_2 - \delta\alpha + o(\delta) \leq \|\lambda(\mathbf{x} + \delta e_d)\|_2 \leq \|\lambda(\mathbf{x})\|_2 + \delta\alpha + o(\delta) \} \quad (4.80)$$

where e_d is the d -th standard basis vector.

This is a bound to the distance by which the functional value can move for a small perturbation in the input. A solution to the parametrized complementarity problem in (4.5) can be viewed as a function from the space of parameter tuples to the solution of the problem. The above definition talks about bounding the change in the solution for a small change in the parameters. The next theorem shows how the total linear sensitivity can be calculated from the linear approximation matrix \mathcal{T} derived earlier.

Theorem 65. Suppose we know, $G \in \mathbb{R}^{n \times m}$ such that $G_{ij} = \frac{\partial f_i(\mathbf{x})}{\partial x_j}$, then $\beta_d = \sqrt{(\sum_{i=1}^n G_{id}^2)}$

Proof. By definition, for some admissible d ,

$$\lambda(\mathbf{x} + \delta e_d) = \lambda(\mathbf{x}) + \delta G e_d + o(\delta) \quad (4.81)$$

$$\Rightarrow [\lambda(\mathbf{x} + \delta e_d)]_i = [\lambda(\mathbf{x})]_i + \delta G_{id} + o(\delta) \quad (4.82)$$

$$\|\lambda(\mathbf{x} + \delta e_d)\|_2 \leq \|\lambda(\mathbf{x})\|_2 + \|\delta G_{\cdot d}\|_2 + \|o(\delta)\|_2 \quad (4.83)$$

$$= \|\lambda(\mathbf{x})\|_2 + \delta \sqrt{\left(\sum_{i=1}^n G_{id}^2 \right)} + o(\delta) \quad (4.84)$$

where $G_{\cdot d}$ is the d -th column of G . Also we have from (4.82) for sufficiently small δ ,

$$\|\lambda(\mathbf{x} + \delta e_d)\|_2 \geq \|\lambda(\mathbf{x})\|_2 - \|\delta G_{\cdot d}\|_2 + \|o(\delta)\|_2 \quad (4.85)$$

$$= \|\lambda(\mathbf{x})\|_2 - \delta \sqrt{\left(\sum_{i=1}^n G_{id}^2\right)} + o(\delta) \quad (4.86)$$

■

The above theorem proves that the \mathcal{T} matrix obtained in (4.61) is sufficient to approximate the total linear sensitivity. The following result suggests how the total linear sensitivity can approximate the total variance in the output variables.

Theorem 66. *Given a function $\lambda : \mathbb{R}^m \mapsto \mathbb{R}^n$ taking random inputs and β_d , the increase in the total uncertainty in the output, i.e., the sum of variances of the output variables, for a small increase of the variance of an input parameter, σ_d^2 of \mathbf{x}_d is approximated by $\beta_d^2 \sigma_d^2$.*

Proof. Let E_d be the matrix of size $m \times m$ with zeros everywhere except the d -th diagonal element, where it is 1. Given $C = \text{cov}(\mathbf{x}(\omega))$ where $x(\omega)$ is the random input for the function λ , for a small perturbation σ^2 in the variance of \mathbf{x}_d , the covariance of $\lambda(\mathbf{x})$ changes as follows.

$$C^* \approx (\nabla_{\mathbf{x}} \lambda) C (\nabla_{\mathbf{x}} \lambda)^T \quad (4.87)$$

$$C^* + \Delta C^* \approx (\nabla_{\mathbf{x}} \lambda) (C + \sigma^2 E_d) (\nabla_{\mathbf{x}} \lambda)^T \quad (4.88)$$

$$= C^* + \sigma^2 (\nabla_{\mathbf{x}} \lambda) E_d (\nabla_{\mathbf{x}} \lambda)^T \quad (4.89)$$

$$[\Delta C^*]_{ij} \approx \sigma^2 [\nabla_{\mathbf{x}} \lambda]_{id} [\nabla_{\mathbf{x}} \lambda]_{jd} \quad (4.90)$$

$$\sum_{i=1}^n [\Delta C^*]_{ii} \approx \sigma^2 \beta_d^2 \quad (4.91)$$

which is the total increase in variance. The off-diagonal terms do not affect the total uncertainty in the system because, the symmetric matrix C can be diagonalized as $Q D Q^T$, where Q is an orthogonal matrix, and the trace is invariant under orthogonal transformations. ■

With the above result, we can determine the contribution of each input parameter to the

total uncertainty in the output. Once the parameter which contributes the most to the output uncertainty is identified, efforts can be made to get a more accurate estimate of the parameter.

4.2.4 Application to optimization

We now show how the method explained in [Algorithm 4](#) can be applied to approximate the covariance of the solution of an unconstrained optimization problem or an optimization problem with only equality constraints.

To start with, we assume conditions on the differentiability and convexity of the objective function.

Assumption A8. *The objective function $f(\mathbf{x}; \theta)$ is strictly convex in \mathbf{x} and is twice continuously differentiable in \mathbf{x} and θ .*

In the theorem below, we approximate the covariance of the decision variables of a convex optimization with uncertainties in the linear term and with only linear equality constraints.

Theorem 67. *With [Assumption A8](#) holding, the covariance of the primal and dual variables at the optimum of the problem,*

$$\underset{\mathbf{x}}{\text{Minimize}} \quad f(\mathbf{x}; \theta) = g(\mathbf{x}) + c(\theta)^T \mathbf{x} \quad (4.92)$$

$$\text{subject to } A\mathbf{x} = b(\theta) \quad (\mathbf{y}) \quad (4.93)$$

where $\theta = \theta(\omega)$ are random parameters with covariance C , is first-order approximated by $\mathcal{T}C\mathcal{T}^T$ where

$$\mathcal{T} = \begin{pmatrix} \nabla_{\mathbf{x}}^2 g(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_{\theta} c(\theta) \\ \nabla_{\theta} b(\theta) \end{pmatrix} \quad (4.94)$$

Proof. For the given optimization problem, because of [Assumption A8](#) and linear independence constraint qualification (LICQ), the KKT conditions are necessary and sufficient for optimality.

The KKT condition satisfied at a solution $(\mathbf{x}^*, \mathbf{y}^*)$ for the problem are given by

$$\nabla_{\mathbf{x}}g(\mathbf{x}^*) + c(\theta) + A^T \mathbf{y}^* = 0 \quad (4.95)$$

$$A\mathbf{x}^* = b(\theta) \quad (4.96)$$

for some vector \mathbf{y} so that the equation is well defined. Suppose from there, θ is perturbed by $\Delta\theta$, we have

$$\nabla_{\mathbf{x}}g(\mathbf{x}^*) + c(\theta + \Delta\theta) + A^T \mathbf{y}^* \approx \nabla_{\theta}c(\theta)\Delta\theta \quad (4.97)$$

$$A\mathbf{x}^* - b(\theta + \Delta\theta) \approx -\nabla_{\theta}b(\theta)\Delta\theta \quad (4.98)$$

Now we need to find $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$ such that

$$\nabla_{\mathbf{x}}g(\mathbf{x}^* + \Delta\mathbf{x}) + c(\theta + \Delta\theta) + A^T(\mathbf{y}^* + \Delta\mathbf{y}) \approx 0 \quad (4.99)$$

$$A(\mathbf{x}^* + \Delta\mathbf{x}) - b(\theta + \Delta\theta) \approx 0 \quad (4.100)$$

$$\nabla_{\mathbf{x}}^2g(\mathbf{x}^*)\Delta\mathbf{x} + A^T\Delta\mathbf{y} \approx \nabla_{\theta}c(\theta)\Delta\theta \quad (4.101)$$

$$A\Delta\mathbf{x} \approx -\nabla_{\theta}b(\theta)\Delta\theta \quad (4.102)$$

The above conditions can be compactly represented as

$$\begin{pmatrix} \nabla_{\mathbf{x}}^2g(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \end{pmatrix} = \begin{pmatrix} \nabla_{\theta}c(\theta) \\ -\nabla_{\theta}b(\theta) \end{pmatrix} \Delta\theta \quad (4.103)$$

If A has full rank, then the above matrix is non-singular. So the change in the decision variables \mathbf{x} and the duals \mathbf{y} can be written as a linear transformation of the perturbation in the random parameters. And we now have

$$\text{cov} \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \end{pmatrix} = \mathcal{T} \text{cov}(\theta) \mathcal{T}^T \quad (4.104)$$

$$\mathcal{T} = \begin{pmatrix} \nabla_{\mathbf{x}}^2 g(\mathbf{x}) & A^T \\ A & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_{\theta} c(\theta) \\ \nabla_{\theta} b(\theta) \end{pmatrix} \quad (4.105)$$

■

In the theorem below, we show that the method suggested is accurate (i.e., has zero error) for an unconstrained quadratic optimization problem with uncertainty in the linear term.

Theorem 68. *For an optimization problem with uncertainty of objectives of the form,*

$$f(\mathbf{x}; \theta) = \frac{1}{2} \mathbf{x}^T G \mathbf{x} + \theta(\omega)^T \mathbf{x} \quad (4.106)$$

the approximation method has zero error. In other words, the obtained covariance matrix is exact.

Proof. For the problem to be well-defined, let $G \in \mathbb{R}^{n \times n}$ and $\theta \in \mathbb{R}^n$. This makes $\nabla_{\mathbf{x}\theta}^2 f(\mathbf{x}; \theta) \in \mathbb{R}^{n \times n}$.

$$\nabla_{\mathbf{x}} f(\mathbf{x}; \theta) = G \mathbf{x} + \theta(\omega) \quad (4.107)$$

$$\nabla_{\mathbf{x}}^2 f(\mathbf{x}; \theta) = G \quad (4.108)$$

$$[\nabla_{\mathbf{x}\theta}^2 f(\mathbf{x}; \theta)]_{ij} = I \quad (4.109)$$

Due to absence of terms dependent on \mathbf{x} in the last two equations, we have an exact equation,

$$G \Delta \mathbf{x} = \Delta \theta \quad (4.110)$$

Due to the exactness of the above equation, we have

$$\mathcal{T} = G^{-1} \quad (4.111)$$

$$\text{cov}(\Delta \mathbf{x}) = \mathcal{T} \text{cov}(\Delta \theta) \mathcal{T}^T \quad (4.112)$$

with no error.

■

4.2.5 Application to a general oligopoly market

We now present an example of a complementarity problem in a natural gas oligopoly and show how the methods developed in this work can be applied.

Problem Formulation and results

Consider k producers competitively producing natural gas in a Nash-Cournot game. Let us assume the unit costs of production are $\gamma_i(\omega)$, $i \in \{1, \dots, k\}$. We assume these are random variables. Also, let us assume that the consumer behavior is modeled by a linear demand curve $P(\tilde{Q})$ as follows.

$$P = a(\omega) + b(\omega)\tilde{Q} \quad (4.113)$$

where P is the price the consumer is willing to pay, \tilde{Q} is the total quantity of the natural gas produced and $a(\omega) > 0$, $b(\omega) < 0 \forall \omega \in \Omega$. Suppose the producers are competitively producing natural gas in a Nash-Cournot game, then the problem that each producer solves is as follows.

$$\text{Producer } i : \quad \text{Maximize} \left(a + b \left(\sum_{j=1}^k Q_j \right) \right) Q_i - \gamma_i Q_i \quad \text{s.t. } Q_i \geq 0 \quad (4.114)$$

where Q_i are the quantities produced by each producer. The KKT conditions of this optimization problem can be written as the following complementarity problem to obtain a Nash equilibrium [47, 77].

In this formulation, a, b, γ_i correspond to θ and Q_i correspond to \mathbf{x} in (4.5) with $\mathcal{I} = \{1, 2, \dots, k\}$.

$$\mathbf{F}_i(\mathbf{Q}) = \gamma_i - a - b \left(\sum_{j=1}^k Q_j \right) - b Q_i \quad (4.115)$$

In the current numerical example, let us consider a duopoly where $k = 2$. Let

$$\mathbb{E} \begin{pmatrix} \gamma_1 & \gamma_2 & a & b \end{pmatrix}^T = \begin{pmatrix} 2 & 1 & 15 & -1 \end{pmatrix}^T \quad (4.116)$$

Solving the complementarity problem deterministically with the above parameter values using the PATH algorithm [57], we get Q_1 and Q_2 to be 4 and 5 respectively. We use the C-function $\psi_{min}(x, y) = \min(x, y)$ for this example to get

$$\mathcal{M} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \mathcal{N} = \begin{pmatrix} 1 & 0 & -1 & -13 \\ 0 & 1 & -1 & -14 \end{pmatrix} \quad (4.117)$$

Now we have from (4.61)

$$\mathcal{T} = \mathcal{M}^{-1}\mathcal{N} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & -12 \\ -1 & 2 & -1 & -15 \end{pmatrix} \quad (4.118)$$

Having obtained \mathcal{T} , we attempt to get insight on how uncertainties in various input parameters propagate through the model causing uncertainty in the equilibrium quantities. If we assume that all these parameters, viz. γ_1, γ_2, a, b have a 10% coefficient of variation and are all uncorrelated, then the covariance matrix of the input is

$$C_1 = \begin{pmatrix} 0.04 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 2.25 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix} \quad (4.119)$$

Then the covariance matrix of the solution would be

$$C_1^* = \mathcal{T}C_1\mathcal{T}^T = \begin{pmatrix} 0.4289 & 0.4389 \\ 0.4389 & 0.5089 \end{pmatrix} \quad (4.120)$$

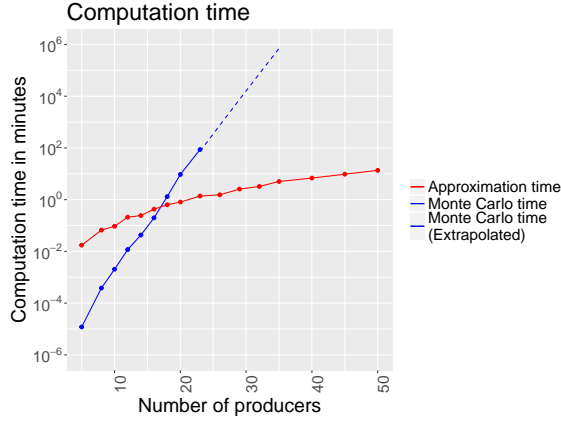
The standard deviation of the produced quantities are $0.65(= \sqrt{0.4289})$ and $0.71(= \sqrt{0.5089})$ respectively. The produced quantities also have about 95% positive correlation as an increase in demand will cause both producers to produce more and a decrease in demand will cause both producers to produce less.

If we assume that we have perfect knowledge about the demand curve, and if the uncertainty is only in the production costs, then the new parameter covariance C_2 has the third and fourth diagonal term of C_1 as zero. In such a scenario, we would expect the decrease in the quantity of production of one player to cause an increase in the quantity of production of the other and vice versa since one producer would lose market share to the other. We can see that by computing the covariance of the solution as $\mathcal{T}C_2\mathcal{T}$. The solution thus obtained shows that the produced quantities are negatively correlated with a correlation of -85% . The uncertainties in the produced quantities are 3% and 2% respectively of the quantity produced by each producer. We also note that the variances are smaller now, as we no longer have uncertainties stemming from the demand side of the problem.

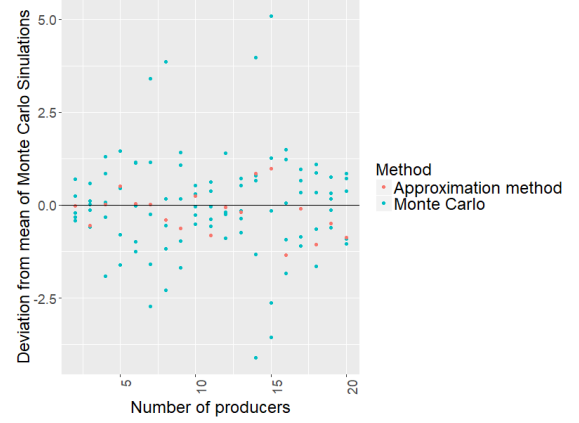
Now if we assume a more realistic scenario of the production costs being correlated (60% correlation), then we note that the produced quantity are negatively correlated with -62% correlation. The standard deviations in the produced quantities have also dropped to about 2.9% and 1.2% of the produced quantities. Thus we not only obtain insight about the uncertainties in the output, but also the correlation between the output parameters. From an energy market policy maker's perspective, this is crucial information as it helps in identifying the regions where production/consumption/price/flows increase/decrease simultaneously and where they change asynchronously. Now we calculate the sensitivity of each of the input parameters to identify the parameter that causes maximum uncertainty in the output. The values for β for each of the four parameters γ_1, γ_2, a, b are calculated below.

$$\beta = \frac{1}{3} \begin{pmatrix} \sqrt{5} & \sqrt{5} & \sqrt{2} & \sqrt{369} \end{pmatrix}^T = \begin{pmatrix} 0.745 & 0.745 & 0.471 & 6.40 \end{pmatrix}^T \quad (4.121)$$

Thus we see that the solution is more sensitive to the slope of the demand curve than to say production cost. Strictly speaking, this says, a unit increase in uncertainty (variance) of the slope



(a) Run time comparison



(b) Error comparison

Figure 4.3: Computational experiments comparing Monte-Carlo methods and First-order approximation method

of the demand curve will be magnified about 41 times (6.4^2) in the output uncertainty. However, a unit increase in the uncertainty of production cost will increase the uncertainty in the equilibrium only by 0.556 units.

Computational Complexity

We used a Monte-Carlo based method as a comparison against our approximation method to compute covariance in the decision variables. To achieve this, we modeled the oligopoly complementarity problem mentioned in (4.115) varying the number of players, and hence the number of random parameters and the decision variables. For the Monte-Carlo simulation based approach, symmetrically balanced stratified design [130] is used with each dimension divided into two strata. With an increasing number of random parameters and equilibrium variables, Monte-Carlo methods become increasingly inefficient as the number of simulations required grows exponentially. A comparison of the time taken in an *8GB RAM 1600 MHz DDR3 2.5GHz Intel*

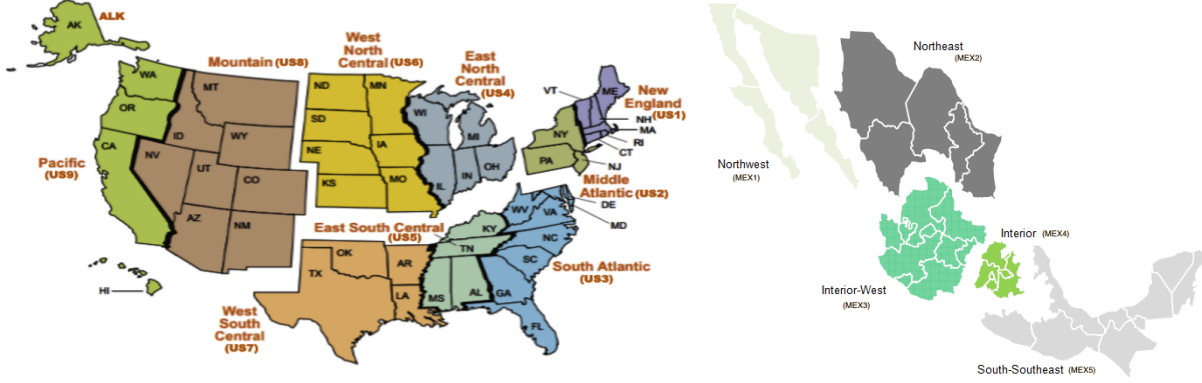


Figure 4.4: Regional disaggregation of United States and Mexico. Source: [3] and U.S. Energy Information Administration <http://www.eia.gov/todayinenergy/detail.php?id=16471>

Core i5 processor to solve the above oligopoly problem with varying number of players is shown in Figure 4.3a. Despite developments in algorithms to solve complementarity problems, the said exponential growth in the number of sample points required in a Monte-Carlo based approach deters the computational speed. A problem with as few as 25 uncertain variables takes about 2 hours to solve and one with 30 uncertain variables takes about seven days to solve using Monte-Carlo based approaches while it takes few seconds to minutes in the first-order approximation method. Figure 4.3b compares the error between 5 rounds of Monte Carlo simulation and the first-order approximation method.

4.2.6 Application to North American natural gas market

In Mexico, motivation to move from coal to cleaner energy sources creates an increasing trend in natural gas consumption. The United States is expected to become a net exporter of Natural gas by 2017 [3] especially due to the continuous increase in exports to Mexico [62]. The North American Natural Gas Model (NANGAM) developed in [69] analyzes the impacts of cross-border trade with Mexico. NANGAM models the equilibrium under various scenarios by competitively

maximizing the profits of producers and pipeline operators and the utility of consumers. The formulation leads to a complementarity problem. The model also uses the Golombek function [80] in the supply function to model the increase in the marginal cost of production, when producing close to capacity. This makes the complementarity problem into a nonlinear one.

In this model, which is motivated by NANGAM, we have disaggregated United states into 9 census regions and Alaska [3]. Mexico is divided into 5 regions. A map showing this regional disaggregation is shown in Figure 4.4. Further, Canada is divided into two zones, Canada East and Canada West. The model has 13 producers, 17 consumers, 17 nodes and 7 time-steps. This amounts to $12,047^2$ variables (primal and dual) and 2023 parameters. A complete description of the model is given in Appendix B. The gradient matrix of the complementarity function would contain $12,047^2$ elements and a Hessian matrix will have 12047^3 elements which is more than 1700 trillion floating point variables. We need efficient methods to handle these large objects. We observe, however, that the dependence of each component of the complementarity function is limited to few variables, thus making the gradient matrix sparse. Efficient sparse matrix tools in scipy [96] are used along with a python class we specially built to handle a sparse multi-dimensional array. The details of the implementation are presented in Appendix C.

This model is calibrated to match the region-wise production and consumption data/projections by adjusting the parameters of the demand curve, the supply curve, and the transportation cost. The source for the projected numbers is the same as the ones in Table 2 of [69]. The parameters of the demand curve were chosen in such a way that an elasticity of 0.29 is maintained at the solution to be consistent with [61].

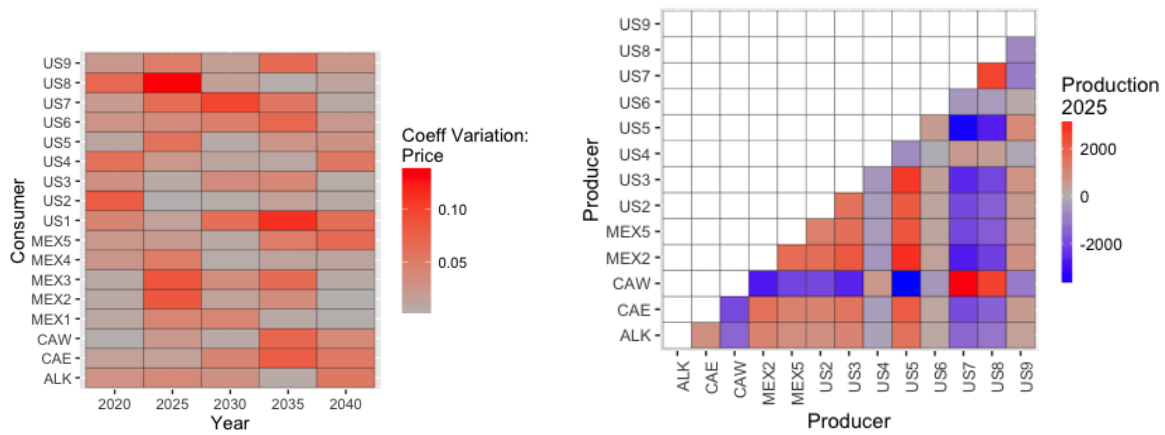


Figure 4.5: Coefficient of variation in Price and Covariance of Produced quantity

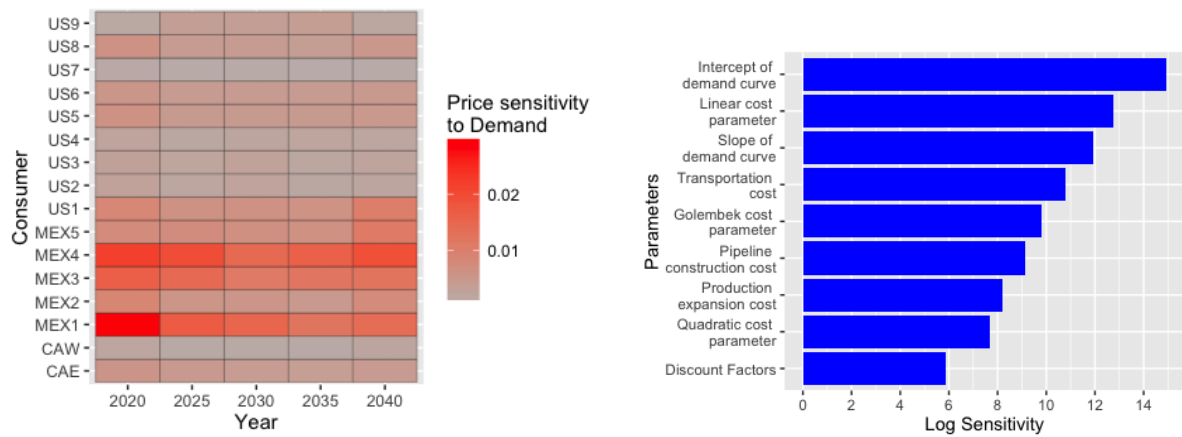


Figure 4.6: Sensitivity of the solution to the parameters

Covariance Matrix Calibration

We used the method developed in [Algorithm 4](#) to understand the propagation of uncertainty in the model. The covariance for each parameter across years is obtained by fitting a Wiener process to the parameter value. This is chosen to mimic the Markovian and independent increment properties of market parameters. Thus we have for any parameter

$$d\theta(t) = d\mu_\theta(t) + \sigma_\theta dB(t) \quad (4.122)$$

where μ_θ is calibrated, σ_θ is chosen to be 1% of the average value of μ_θ in the analyzed period and $B(t)$ is the standard Brownian motion. The diffusion parameter σ_θ is assumed to be independent of time. Additionally to understand the effect of greater uncertainty in the US census region US7, which contains Texas and accounts for about 40% of the total production in the continent, the parameters of production cost are assumed to have 5 times the variance than in any other region.

Results

The deterministic version of the problem is solved using the PATH algorithm [\[57\]](#) by assuming a mean value for all random parameters. With that solution as the initial point, we used the algorithm in [\[95\]](#) to solve the stochastic version of the problem and obtain the solution to the SCP. Following this, [Algorithm 4](#) was applied and the \mathcal{T} matrix defined in [\(4.61\)](#) is obtained by solving the linear system of equations using a Moore Penrose pseudoinverse [\[87\]](#). In the following paragraph, we discuss some of the results obtained in this study.

The heat map on the left of [Figure 4.5](#) shows the coefficient of variation (std dev divided by mean) in consumer price in each year caused by the uncertainty in parameters as mentioned in [Section 4.2.6](#). We notice that this uncertainty in production costs of US7 caused relatively small uncertainties in the consumer price. This is partially due to the availability of resources in US8 and

Canada West to compensate for the large uncertainty in Texas (US7). The fact that it is actually US8 and Canada West that compensate for this uncertainty is known by looking at the covariance plot on the right of [Figure 4.5](#) which shows a large correlation between US7 and US8 and also between US7 and CAW.

[Figure 4.6](#) shows the sensitivity of the solution to various input parameters. The graph on the left shows the sum total change in uncertainty in price for a 1% fluctuation in the demand curve of various consumers. We notice that the price is particularly sensitive to changes in demand in Mexico. We also note that fluctuations in demand at nodes where production facilities are not available (MEX1, MEX3, MEX4) cause greater uncertainty in price. This is because, for regions with a production facility in the same node, the production facility produces more to cater the demand at that node and there is little effect in the flows and in the prices at other nodes. However, a perturbation to the demand at a node with no production unit causes the flows to alter to have its demand catered. This affects natural gas availability elsewhere and causes larger fluctuations in price. The tornado plot on the right of [Figure 4.6](#) sorts the parameters in decreasing order of their effect on the uncertainty of the solution. Noting that it is plotted in log scale, we understand that uncertainties in the intercept of the demand curve affect the equilibrium the most. Among the supply parameters, the linear cost parameter causes maximum fluctuation in the output.

4.2.7 Conclusion and future work

In this work, we developed a method to approximate the covariance of the output of a large-scale nonlinear complementarity problem with random input parameters using first-order metrics. We extended this method to a general unconstrained optimization problem. We then developed sensitivity metrics for each of the input parameters quantifying their contribution to the uncertainty

in the output. We used these tools to understand the covariance in the equilibrium of the North American Natural Gas Market. The method gave insights into how production, consumption, pipeline flows, prices would vary due to large uncertainties. While the variances identified the regions that are affected the most, the covariance gave information about whether the quantity will increase or decrease due to perturbation in the input. We also obtained results on the sensitivity of price uncertainty to demand uncertainty in various nodes. We then quantified the contribution of each input parameter to the uncertainty in the output. This, in turn, helps in identifying the regions that can have large impacts on equilibrium.

We note that the method is particularly useful for large-scale nonlinear complementarity problems with a large number of uncertain parameters to make Monte-Carlo simulations or methods involving scenario trees intractable. It is robust in approximating the solution covariance for small uncertainty in the inputs. It is also good in quantifying the sensitivity of the output (and its variance) to the variance of various input parameters. However since all the above are obtained as an approximation based on first-order metrics, there is a compromise in the accuracy if the variances of the input are large. The method works the best for problems involving a large number of decision variables and random parameters with small variance.

We foresee expanding this work by using progressively higher-order terms of the Taylor series to capture the nonlinearities more efficiently. To ensure computational feasibility, this would typically require us to have stronger assumptions on the sparsity of the Hessian and the higher-order derivatives. This will also require knowledge/assumptions about higher-order moments of the random parameters.

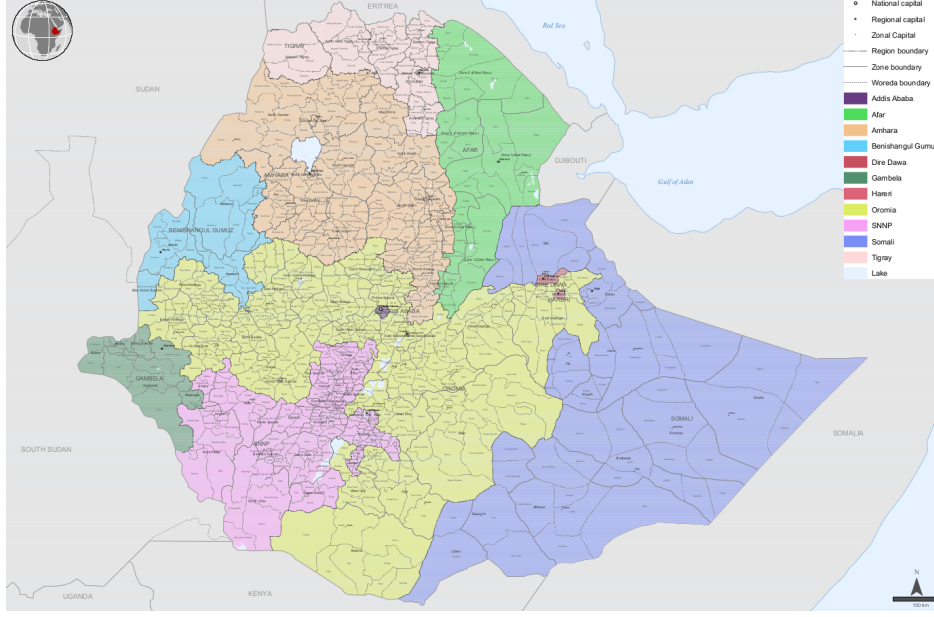


Figure 4.7: Administrative regions in Ethiopia. Credit: UN Office for the Coordination of Humanitarian Affairs (UN-OCHA)

4.3 Ethiopian food market

4.3.1 Introduction

Ethiopia is an east African country with a population of about 105 million [4, 39] and a GDP of 72 billion US dollars [4]. Over 70% of the labor force is dependent on agriculture for their livelihood [4] and agriculture contributes to about 40% of the GDP of the country [4]. Moreover, about 60% of all exports are agricultural products [115] leading to extensive studies on the country's agricultural sector. However, agriculture in Ethiopia is vulnerable to changes in climate patterns and droughts since the country is dominated by rainfed agriculture and the irrigated farmland area in Ethiopia is less than 5% of the cultivated land [13]. This led to an extensive study of the climate shocks on crop production in the literature [13, 14, 113].

In addition to climatic shocks, the country's agriculture and food supply chain is not immune to

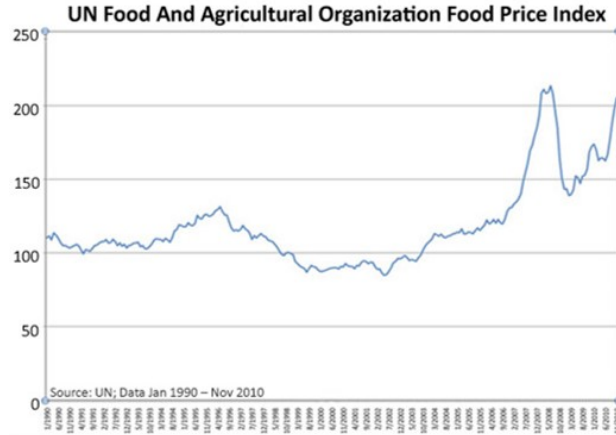


Figure 4.8: Global food price time series. Credit: Food and Agriculture Organization

other exogenous shocks. The 2008 global food price inflation (see [Figure 4.8](#)) caused deep distress to the country's food sector. Alem and Söderbom [6], Kumar and Quisumbing [100] quantify the impact of the price hike in terms of changes in food consumption pattern and changes in nutrient intake through a survey. The adverse effect of such a global change motivated the Ethiopian government to ban the export of food grains, in particular, teff indefinitely. This was done with an intention to ensure more affordable domestic prices. Woldie and Siddig [140] analyze the impacts of the ban on the country's macroeconomy under this scenario. While predicting that the policy will indeed fulfill its goal of reducing domestic food prices, it comes at a cost of social welfare, quantified post facto, at about 148 million US dollars. Further, the government policy was also studied extensively in Sharma [129] which criticized the government policy comparing it against similar policies in other countries and the harmful effects thereon. Instead, the author suggests other alternatives for the government including various tax regimes, price floors for exports, government to government sales.

While the existing literature quantifies the effects of past shocks in global food prices and government policies, to the best of our knowledge, there are few resources that can analyze the

distributed impacts of future shocks in local regions under different government policies. In this work, we address the variations in regional teff markets, transport patterns, and stakeholder revenue across disaggregated regions in Ethiopia due to the teff-export ban policy. The specific interest towards teff is motivated by the fact that teff has been predicted by some to become a new “super-grain” [48, 122], with dramatically increased international demand. Our analysis complements the existing analyses in Sharma [129], Woldie and Siddig [140] in providing a detailed understanding of the *regional* effects of such shocks, along with comparing the effects of the shocks under possible futuristic demand.

Given this global scenario, we address the following questions regarding potential relaxation or removal of teff export restrictions:

1. How do regional microeconomic market indicators change under different policies?
2. How does the production and consumption pattern change across the country, and which regions in the country are impacted the most?
3. Which regional stakeholders benefit by the increased revenue from exports?

To answer these questions, we present an integrated partial-equilibrium model, Food Distributed Extendible COmplementarity model 2 (Food-DECO 2), an extension of the model in [14]. The model is a partial-equilibrium model, that models the behavior of multiple stakeholders in a market, non-cooperatively maximizing their objective (typically profit or utility) with detailed modeling of the agro-economic properties of the regions in the country and the food markets therein. This detailed modeling enables us to present regionalized effects of the shocks to the country’s food market.

In this work, we run the model for several scenarios of global shocks that affect demand for teff

and compare the performance of different government policy under each of the scenarios. Such an analysis helps compare policy under various stimuli and provide quantifiable differences in regional welfare.

The rest of the chapter is divided as follows. We use [Section 4.3.2](#) to describe the features and enhancements of the DECO2 model. [Section 4.3.3](#) present the scenarios we analyze in detail and explains the method employed in implementing these scenarios and [Section 4.3.4](#) contains the results and discussions.

4.3.2 Model description

The Food-Distributed Extendible Complementarity model (DECO2) is an integrated partial-equilibrium model that models the food supply chain in Ethiopia. Partial-equilibrium models find an equilibrium among the micro-economic variables, while assuming macroeconomic variables (i.e., population, GDP) as constant, more popularly known in economics as *ceteris paribus*. Modelers follow this approach typically to capture more granular detail than *general equilibrium models*, which factor all macroeconomic variables but don't explicitly model regional markets and infrastructure. Partial-equilibrium models are more suited for analyzing localized changes under isolated shocks, and not to project future macroeconomic trends.

In DECO2, we compute the equilibrium resulting from the interaction of five types of aggregated stakeholders in the model namely:

1. Crop producers
2. Livestock raisers
3. Food distribution operators

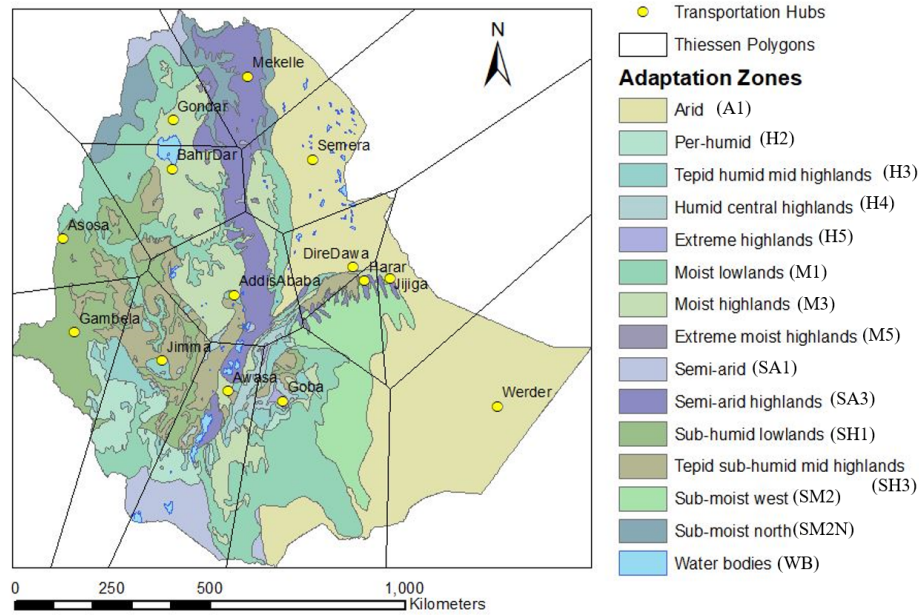


Figure 4.9: Adaptation zones of Ethiopia

4. Food storage operators or warehouses
5. Food consumers

Each stakeholder competitively maximizes her own objective under perfect competition. The model disaggregates Ethiopia into multiple regions which accounts for variability in climate as well as soil fertility patterns across the country. This disaggregation helps in explicitly modeling food production and consumption within the country. Given that regions with similar agro-economic properties are not necessarily contiguous, we also have markets across the country where the producers can participate and sell their produce and intra-national transport occurs between the markets.

Spatial disaggregation

We use two types of spatial disaggregation in the model, namely disaggregation by adaptation zone to represent production and consumption and by regions centered with selected transportation hubs to model regional markets and transportation. The 14 adaptation zones divide the country into regions of similar weather patterns and soil fertility patterns, so that in our model, any crop will have the same yield across a single adaptation zone. These zones have been defined by the Government of Ethiopia as a framework for climate resilience efforts. These zones are defined with production rather than consumption in mind but given the dominance of smallholder subsistence agriculture in Ethiopia, we use the adaptation zones as the unit for modeling consumption as well. However, the adaptation zones are not necessarily a contiguous stretch of land, and they can spread over long distances geographically. For this reason, we cannot use adaptation zones to model food transport. For this purpose, we divide the country into 15 transportation hubs or markets and assume that all transport occurs between pairs of these markets. The markets were chosen to ensure that they were both reasonably spread across the country and correspond to highly populated cities in the country. We use the markets as seeds to a *Voronoi tessellation*, such that the country is partitioned into Thiessen polygons, each of which contains exactly one market (see [Figure 4.9](#)). The Thiessen polygons have the property that the market contained in the polygon is the closest market to every point within the polygon [137]. We assume that any crop producer in the country sells in the market in their Thiessen polygon. This is equivalent to saying that the crop producer sells in the market closest to them geographically. This assumption is reasonable since we have divided the country into only 15 polygons; we are not literally claiming that producers go to the closest village market, but rather that they sell into the market of their general region.

Exports from Ethiopia are modeled by adding an “external” node to the collection of

transportation hubs. The prices and demand in the external node are set by global prices for food. Any export from Ethiopia is sent to the external node and any import to Ethiopia comes in through the external node, which is connected to other nodes via the national capital, Addis Ababa.

Time steps

DECO2 solves for equilibrium in semi-annual time steps, each step corresponding to a cropping season. To ensure realistic modeling of responses to shocks in climate and global economy, we solve the model on a year-by-year basis, without information about future years. On the other hand, to remove excessive model short-sight, two additional years are solved in each iteration and then dropped. We call this the “rolling-horizon process” and refer readers to the paper by Sethi and Sorger [127] for a more rigorous analysis of the approach.

Within each year, we explicitly model the two cropping seasons of Ethiopia. The *meher* season, which relies on the summertime *kremt* rains (primarily June-September), and the springtime *belg* season, which relies primarily on March-April rains. The meher season is the primary cropping season, in which more than 70% of all food is produced. We now discuss the six stakeholders we model in DECO2. They are all profit maximizing stakeholders, aside from the consumer who maximizes utility. The model assumes a non-cooperative game played between the stakeholders under an assumption of perfect competition.

Now we describe the different stakeholders in the model and their optimization problem. A formal description of these problems and the KKT conditions associated with them are in [Appendix D](#).

Crop producer

We assume that the agricultural land in Ethiopia is used to produce either primary food crops or secondary food crops or cash crops. The primary food crops are teff, sorghum, barley, maize and wheat, the secondary food crops are pulses, vegetables, and fruits, and the cash crops are coffee and oilseeds. The primary crops are grown in about 70% of the total cropping area while the secondary and cash crops are grown in roughly 20% and 10% of the total cropping area respectively. We assume, that there is a representative aggregate crop producer in each adaptation zone who decides the proportion of land allotted for each of the crops. The aggregate crop producer decides this with an objective to maximize her profits, given the knowledge of the yield she could have for the crop while constrained on the total area she can crop. In the zone z , for the crop c , during the cropping season s , in the year y , the problem of the food producer can be written as

$$\text{Maximize}_{A_{zcsy}} : \sum_{c,s,y} \left(\sum_n Q_{znscy}^{F \rightarrow D} \pi_{nscy}^{D \rightarrow F} - C_{zcsy}^F A_{zcsy} \right) - \text{penalty terms} \quad (4.123)$$

subject to

$$\sum_c A_{zcsy} = A_z^{\text{Total}} \quad (4.124)$$

$$Q_{zcsy}^F = \Upsilon_{zcsy} A_{zcsy} \quad (4.125)$$

$$Q_{znscy}^{F \rightarrow D} = \Psi_{zn} Q_{zcsy}^F \quad (4.126)$$

In this formulation, Q_{zcsy}^F refers to the total quantity produced in the region. The crop producer sells $Q_{znscy}^{F \rightarrow D}$ to a distributor in the market or hub n which fetches a price of $\pi_{nscy}^{D \rightarrow F}$. C_{zcsy}^F stands for the cost of raising and maintaining the crops.

The decision variables are the area that the crop producer allocates for each crop in the adaptation zone. The crop producer decides this to maximize her profits, which is the difference

between the revenue obtained by selling various crops to distributors in different cities or nodes n , and the cost incurred by growing them. We also include a penalty term which penalizes substantial changes in cropping patterns in consecutive years. This happens by subtracting a large value proportional to the difference in cropping patterns between consecutive years, from the objective (which the crop producer wants to maximize). For brevity, we have not detailed the precise form of the penalty term and this can be found in the more formal set of equations in [Appendix D](#). This approach mimics the real-life behavior of crop producers, who are reluctant to drastically change cropping patterns in response to single year fluctuations in climate.

The constraint in [eq. \(4.124\)](#) ensures that the sum of areas allotted for each crop equals the total cropping area in each adaptation zone. The constraint in [eq. \(4.125\)](#) connects the cropped area with the yield of the crop. The yield of a crop, \mathbb{Y}_{zcsy}^F , is calculated using the crop yield model based on an FAO approach described in Allen et al. [7], Doorenbos and Kassam [59]. In summary, using historical data, the model helps us predict the yield of crops (quintals per hectare) under various water conditions, which are affected by climatic patterns and irrigation patterns. The climatic data that goes into yield prediction include meteorological inputs such as temperature, precipitation, humidity, wind, solar radiation and cloud cover. For each growing season, a climate yield factor (CYF) is output by this model. The CYF ranges from 0 to 1, with 0 indicating total crop failure and 1 indicating no water stress.

The constraint in [eq. \(4.126\)](#) decides the proportion of the crop producer's production that goes to each city or node n from the adaptation zone z . Keep in mind that, given that the adaptation zones are just regions of similar agro-economic features, these can be quite disconnected. Regions in different adaptation zones can be geographically clustered together while regions in the same adaptation zone can be scattered across the country. Hence the proportion, Ψ_{zn} , is decided by

the percentage of area in the adaptation zone z that is geographically closest to the node n . The price $\pi_{ncsy}^{D \rightarrow F}$ that the crop producer gets is the price a food distributor at node n is ready to pay for the crop c . We note that there is a single unique price for the crop c at any node n at a time, irrespective of who the seller could be (i.e., the adaptation zone z). Since we do not grade the quality of crop in one region over the other, this is a direct consequence of the economic principle that if there are two different prices at a place and time, the buyer would only buy the crop at a lower price. By the principle, the price, $\pi_{ncsy}^{D \rightarrow F}$, is the so-called “clearing price” or the “equilibrium price” of the crop.

Finally, the quantities in parentheses at the end of each constraint are the dual variables corresponding to the constraints. They quantify the impact of the constraint to the optimization problem. In other words, their value is the proportional increase in the objective value for a marginal relaxation of the constraint.

Livestock producer

The livestock raiser is also a food producer, like the aggregated crop producer. However, in contrast, livestock raisers produce beef and milk in quantities proportional to the number of cattle they raise. We do not attempt to model the climate sensitivity of livestock growth or survival rates in the current version of the model. However, if adverse climate conditions led to a small crop yield, the livestock raiser might slaughter more cattle to raise beef production in a certain year to provide for the food demand. Livestock holdings are thus sensitive to climate via climate’s impact on crops. Their optimization problem is shown below.

$$\text{Maximize}_{Q_{zsy}^C, Q_{zsy}^{sl}} : \sum_{s,y} \left(\sum_{n,\xi} Q_{znsy}^{C \rightarrow D,\xi} \pi_{nsy}^{D \rightarrow C,\xi} - C_{zsy}^c Q_{zsy} \right) \quad (4.127)$$

subject to

$$Q_{zsy}^C = (1 + \text{birth rate} - \text{death rate})Q_{zs(y-1)}^C - Q_{zsy}^{sl} \quad (4.128)$$

$$Q_{zsy}^{sl} \geq kQ_{zsy}^C \quad (4.129)$$

$$Q_{zsy}^C \geq \text{Herd size requirement} \quad (4.130)$$

$$Q_{zsy}^{C,\text{milk}} = \mathbb{Y}_{zsy}^{\text{milk}} Q_{zsy}^C \quad (4.131)$$

$$Q_{zsy}^{C,\text{beef}} = \mathbb{Y}_{zsy}^{\text{beef}} Q_{zsy}^{sl} \quad (4.132)$$

$$Q_{znsy}^{C \rightarrow D, \xi} = \Psi_{zn} Q_{zsy}^{C, \xi} \quad (4.133)$$

In this formulation Q_{zsy}^C refers to the quantity of cattle raised in a year and Q_{zsy}^{sl} refers to the quantity of cattle slaughtered in the year. ξ refers to both beef and milk, the produce from cattle. The objective of the player is to maximize the difference between the revenue from selling milk and beef in various cities indexed by n and the cost of raising cattle.

The first constraint, [eq. \(4.128\)](#) connects the number of cattle from one year to the next. The count varies due to both fertility and mortality of cattle, and due to slaughtering. The mortality rate is typically zero or a very small number, as the player generally slaughters the animal before it dies.

The second constraint, [eq. \(4.129\)](#) ensures that the player slaughters at least a certain fraction of cattle each year, without which they might die of natural causes. In contrast, the third constraint ensures that the player does not slaughter too many animals, thus not being able to maintain the herd size she desires to have.

The next constraints, [eqs. \(4.131\)](#) and [\(4.132\)](#) connect the quantity of milk with the number of animals alive and the quantity of beef with the number of animals slaughtered. Like the crop producer, the livestock producer is also expected to sell her produce to various cities in a fixed

proportion, based on geographical proximity. This is taken care of by the last constraint, [eq. \(4.133\)](#). The prices in the formulation are the clearing prices obtained in the market. This is again, similar to the food crops. We assume that the distributors and the storage operators do not differentiate between the food crops and the animal products in the model. We also assume that the quantity of agricultural and animal products produced is capable of meeting a market demand for food or a food consumption need. This simplification could be modified in future versions of the model.

Distributor and warehouses

The distributor buys agricultural and animal products from crop producers and livestock raisers in the cities across the country. Each city has a unique price for each crop, for exchange between producer and distributor. Keeping in mind that there are different regions which preferentially grow different crops, and that the consumption preference might not coincide with production, the distributors move the agricultural and animal products between the markets in different cities. They act as arbitrageurs taking advantage of the price differential between these markets while incurring the cost of transporting the goods. We fix the cost of transport as a function of time taken to travel in between cities and the distance between them.

The warehouses in each city store agricultural and animal products from the day of harvest to consumption, potentially across seasons or years in case of predicted drought. They also sell the goods to the consumer, acting as the market maker. While they endure a cost of storage, they also charge a markup in prices for the consumers.

Consumer

We assume homogeneous food consumption in each adaptation zone, but not across adaptation zones to account for subsistence-farming-type demand in the model. More precisely, we assume

that preference for choice of food changes based on the regional production due to subsistence farming practices [38], and hence assume similar utility function of food for people in the same adaptation zone.

The utility function is constructed using the price elasticity of various crops across the country. People from the given adaptation zone, who exhibit similar preferences, might be forced to purchase food from different markets in different cities at different prices. Thus, the problem is modeled as follows.

$$\text{Maximize}_{Q_{zcsy}^U} : \sum_{c,s,y} \left(U_{zcsy} (Q_{zcsy}^U) - \sum_n \pi_{ncsy}^{U \rightarrow W} Q_{zncsy}^{W \rightarrow U} \right) \quad (4.134)$$

subject to

$$Q_{zncsy}^{W \rightarrow U} = \Psi_{zn} Q_{zcsy}^U \quad (4.135)$$

$$Q_{zcsy}^U \leq Q_{zcsy}^{\text{export limit}} \quad (4.136)$$

Here $U_{zcsy}(\cdot)$ is the utility function of the consumers in the adaptation zone z . Q_{zcsy}^U is the actual quantity consumed in zone z while $Q_{zncsy}^{W \rightarrow U}$ is the quantity consumed in zone z that is purchased from a warehouse in node n . The objective is to maximize the difference between the utility of consumption and the money paid to obtain the utility. The constraint [eq. \(4.135\)](#) connects the total quantity consumed with the quantity purchased from each warehouse. Note that $\pi_{zncsy}^{U \rightarrow W}$ is not dependent on z , implying a single price for a crop in warehouses irrespective of the consumer. Again, these prices are the market-clearing prices at the warehouse. For simplicity, the utility functions have been assumed to be concave quadratic functions, a standard practice used in previous studies [\[69, 70, 77, 125\]](#).

Market clearing

Besides each stakeholder's optimization problem, we have the so-called market-clearing conditions which are not a part of any stakeholder's optimization problem but connect the optimization problems of different stakeholders. They also have dual variables associated with them, and they represent the price of the commodity that gets cleared in the equations. We have three market clearing equations in the DECO2.

$$\sum_z Q_{zncsy}^{F \rightarrow D} = Q_{ncsy}^{D_b} \quad (\pi_{ncsy}^{D \rightarrow F}) \quad (4.137)$$

$$Q_{ncsy}^{D_s} = Q_{ncsy}^{W_b} \quad (\pi_{ncsy}^{W \rightarrow D}) \quad (4.138)$$

$$Q_{ncsy}^{W_s} = \sum_z Q_{zncsy}^{W \rightarrow U} \quad (\pi_{zncsy}^{U \rightarrow W}) \quad (4.139)$$

The constraint in [eq. \(4.137\)](#) ensures that the total quantity of agricultural and animal products sold by crop producers and the livestock producer in each market n equals the quantity bought by the distributor. The constraint in [eq. \(4.138\)](#) ensures that the quantity sold by the distributor is the quantity bought by the storage operator and the constraint in [eq. \(4.139\)](#) ensures that the quantity stored by the storage operator in market n is purchased by some consumer in one of the adaptation zones z .

Food exports

To model food exports, we have an “external” node that is connected to the national capital, Addis Ababa. The food distributor, under this setting, does not only transport food between the set of cities in Ethiopia, but also to the external node.

The external node allows us to model global demand using a global demand curve, which can

be informed by global price for a commodity and its elasticity. To model specific policy scenarios of export limits, this node also can have a consumption limit, which enables us to model caps on exports.

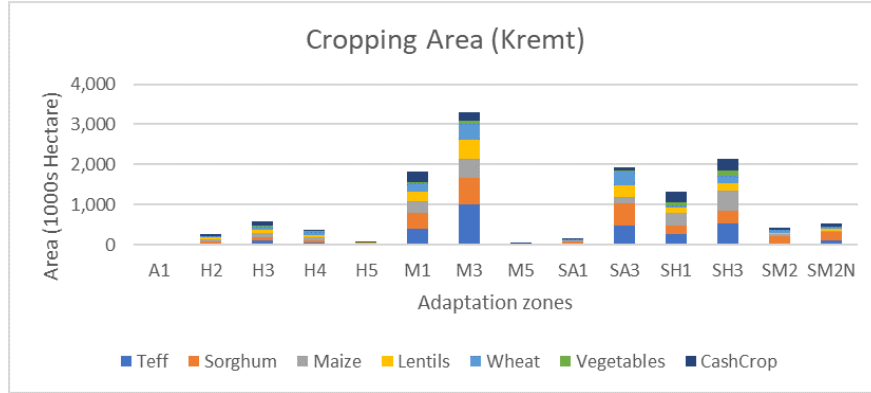
Model limitations and simplifying assumptions

While the model has a detailed description of the agro-economic zones to model crop production and markets to model food trade and transport, like every model, DECO2 has its own set of limitations and simplifying assumptions. Being a partial-equilibrium model, DECO2 assumes invariability of macroeconomic parameters that could potentially affect the variables considered in the model. For example, we do not consider changes in GDP of the country throughout the time-horizon over which the model is solved. This contrasts with the reality where the GDP of Ethiopia has been growing at about 8-10% per annum over the last decade. Similarly, Ethiopia also imports about 15 million quintals of wheat per annum over the last few years. We again assume that these imports stay constant throughout the time horizon.

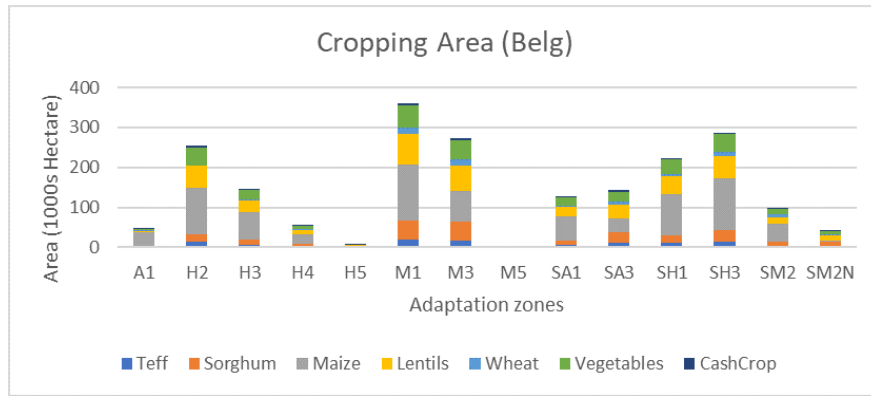
These assumptions on the macroeconomic variables could be unrealistic if we are interested in future macroeconomic trends. However, we observe that the macroeconomic variables affect all the scenarios (including the reference) and shocks uniformly. So, if we are interested in determining the direct impacts of shocks, and evaluating the differences in microeconomic parameters, DECO2 serves as a useful tool to compare policies on a “what-if” basis.

4.3.3 Base case calibration and scenarios

In this section, we discuss the source of data for our model and various scenarios we run our model on and discuss their relevance to Ethiopia.



(a) Cropping Area distribution in Kremt Season

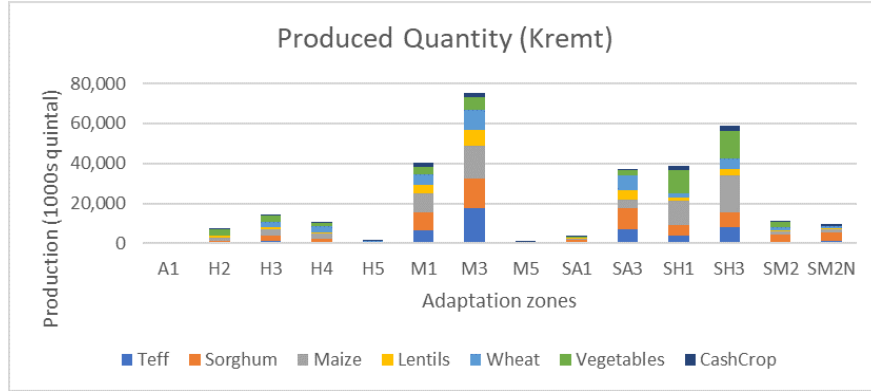


(b) Cropping Area distribution in Belg Season

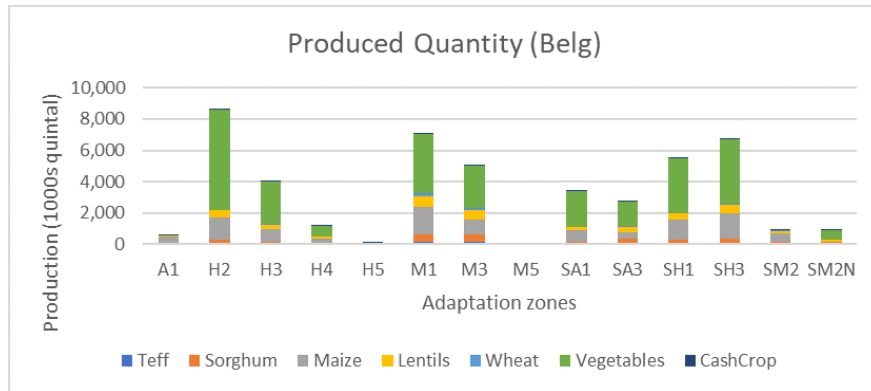
Figure 4.10: Calibration data for cropping area by the crop producer. The details of the adaptation zone abbreviations are in [Figure 4.9](#).

Crop area and yield

Area allotted for various crops in 2015 is available in the report on Area and Production of Major Crops [38] for both kremt and belg season at woreda level. Woredas are administrative units of Ethiopia and the country is divided into 64 administrative zones. The production data is aggregated into production in each adaptation zone. If an administrative zone is completely contained in an adaptation zone, all production in the zone is counted for the adaptation zone. However, if the boundary of an administrative zone cuts a woreda into two (or more) parts, then the production



(a) Production in kremt season

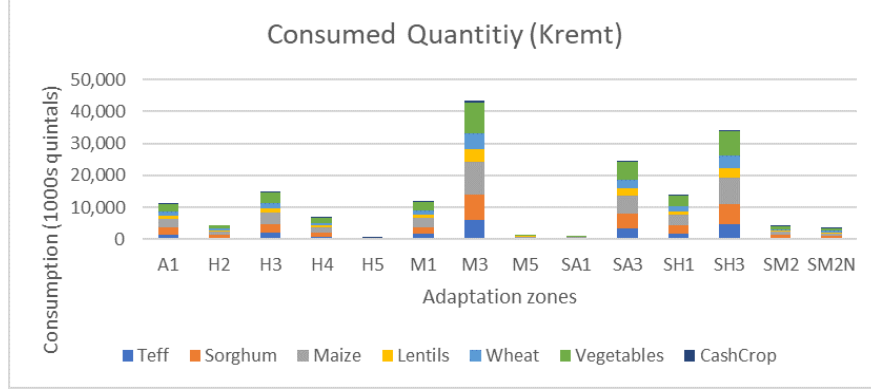


(b) Production in belg season

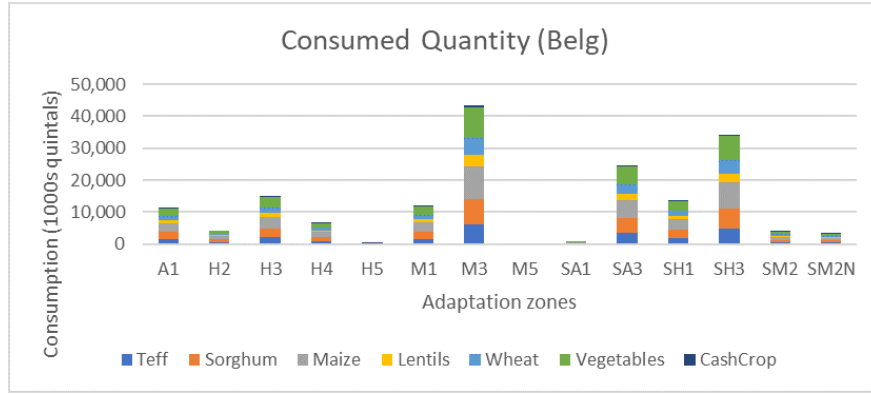
Figure 4.11: Calibration data for quantity produced by the crop producer. The details of the adaptation zone abbreviations are in [Figure 4.9](#).

is divided into the two (or more) adaptation zones proportional to the area each adaptation zone contains.

Over 23% of the total agricultural land is used for teff and about 22% of the land is used for sorghum and barley. Cash crops, which includes oil seeds and coffee, use about 11% of total land in the base case scenario. Teff production the country is about 50 million quintals. Barley and sorghum together amount to about 65 million quintals while maize production is around 75 million quintals, making them the most significant food crops, by production quantity. A summary of the



(a) Consumption in Kremt



(b) Consumption in Belg

Figure 4.12: Calibration data for consumed quantity. The details of the adaptation zone abbreviations are in [Figure 4.9](#).

cropping area and production quantity is shown in [Figure 4.10](#) and [4.11](#).

Consumption

The model is run with the notion that all crops that are produced are consumed. The quantity consumed is assumed to be proportional to the population in each region. The data for the population is obtained from MIT [\[115\]](#). Further, the demand curve for each adaptation zone is calibrated as follows. Price elasticity e , which is defined as the percentage change in consumption

for a unit percent change in price, is obtained from Tafere et al. [133]. The utility function mentioned in Section 4.3.2 is assumed to be a quadratic function of the form $aQ - bQ^2$ for some a and b . This leads to an inverse demand curve given by $\pi = a - bQ$. This along with the equation for price elasticity $e = -\frac{dQ}{d\pi} \frac{\pi}{Q}$, gives the parameters of the utility function as $a = \pi \left(1 + \frac{1}{e}\right)$ and $b = -\frac{1}{e} \frac{\pi}{Q}$. The national average price for all the crops are obtained from Tafere et al. [133] and we assume that the same price is observed in all the markets for calibrating the demand curve.

Transportation

Transportation is modeled between pairs of markets in Ethiopia are the markets are located in a chosen set of 15 cities mentioned earlier. The distances between any pair of cities through road is obtained through [84]. The cost of transportation is proportional to both the quantity of food transported and the distance between markets. Given the production and consumption costs, we let the model decide the optimal transportation between the markets.

Scenarios

In this work, we are interested in comparing the regional changes in market indicators, namely regional food transport, crop producers' revenue, regional consumption and prices within Ethiopia, due to teff-related governmental policies under typical as well as adversarial future global demand. This requires an adequate granularity in the representation of the teff supply-chain infrastructure. To do this, we include detailed modeling of teff production area, yield, transport, markets, storage, and consumption. In addition, it is also necessary to model the effects due to other crops that could share resources with teff. The DECO2 model developed as a part of this work considers all these effects extensively and analyzes policies and intervention strategies.

With respect to the long-standing policy of Ethiopia's cereal export ban, we are interested in

answering the following questions, mentioned earlier in [Section 4.3.1](#):

Compared with the case where the teff ban is still in place and similar global demand for teff,

1. How do regional microeconomic market indicators change under different policies?
2. How does the production and consumption pattern change across the country, and which regions in the country are impacted the most?
3. Which regional stakeholders benefit by the increased revenue from exports?

under the following exogenous shocks:

- (i) When the government allows about 200,000 quintals of teff export per year (or 100,000 quintals of teff export per season), motivated by Abdu [2].
- (ii) When the government allows a higher quantity of exports of up to 2 million quintals of teff each year (or 1 million quintals of teff export per season)
- (iii) When the government allows a fully free market for teff, allowing the market to decide the quantity exported and the price. We run these scenarios under both typical and increased global demand for teff, which indeed was followed by the teff ban by the government. We analyze the results of this study in the next section.

4.3.4 Results

In this section, we will compare how different stakeholders react and benefit under different scenarios of government policies and global demand. First, we compare the additional revenue that the crop producers could potentially get and then we compare domestic average changes in the price of teff under each of these scenarios.

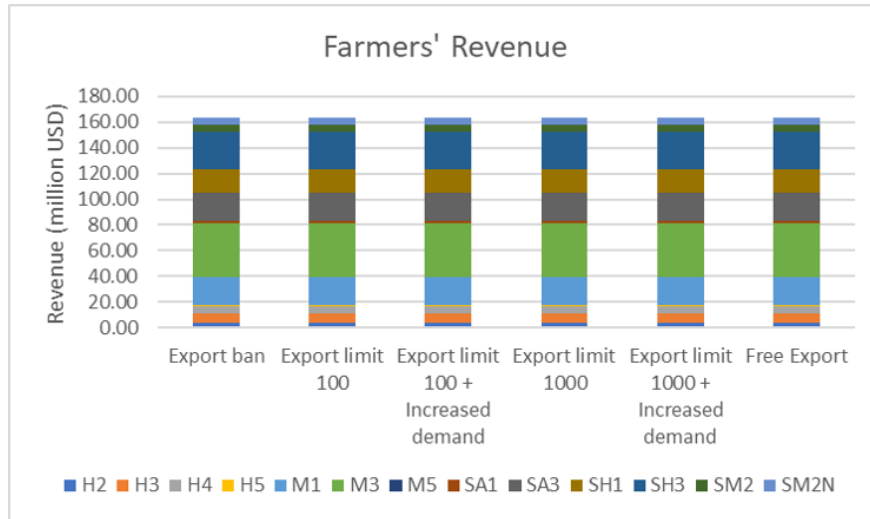
Changes in crop producers' revenue

People who advocate for the removal of teff export ban do so primarily on account of the lost international revenue from teff export [26, 79]. Crymes [48], for example, assumes a direct increase in the revenue and hence the welfare of the crop producers due to increased international revenue. While it is true that the foreign revenues due to teff export range from 0 USD under a complete ban to 15 million USD and 160 million USD under 200,000 quintal per annum export case and 2 million quintal per annum export case, we observe that hardly any revenue reaches the crop producer. We notice, as shown in Figure 4.13a, that the crop producers' revenue, irrespective of their geographical location, remains roughly the same under each of these scenarios. The additional inflow of money from the export is enjoyed almost exclusively by the distributors and storage operators who transport the commodity. So, unless a crop producer is a large-scale stakeholder who can afford to do her own distribution of her production across the country or outside, the increase in her revenue is marginal.

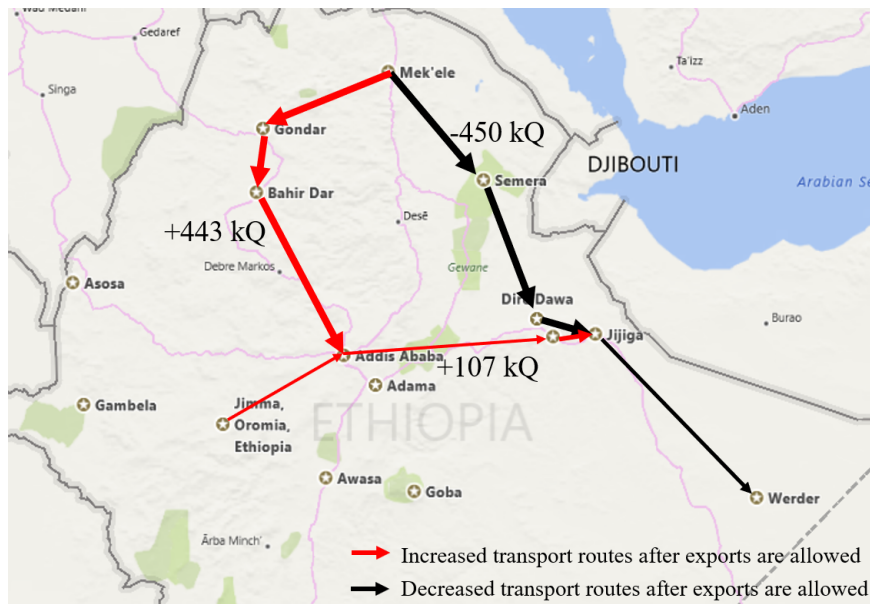
Changes in teff transportation pattern and domestic consumption

Now we analyze the regions that primarily contribute to teff export and the changes in domestic consumption patterns. Teff from the fertile regions in Amhara and Tigray regions contribute to most of the teff exported from the country. This is observable due to the significant changes in transportation patterns of teff in Figure 4.13b. We notice that the teff that should have otherwise gone to Dire Dawa and thereon to Jijiga and Werder is diverted to Addis Ababa, from where export occurs. We also notice that relatively small quantities of teff from the Oromia and SNNPR regions of Ethiopia is sent for export.

Due to this effect, some markets sell significantly less teff than before (Figure 4.14; note that the



(a) Aggregated crop producers' revenue under different scenarios

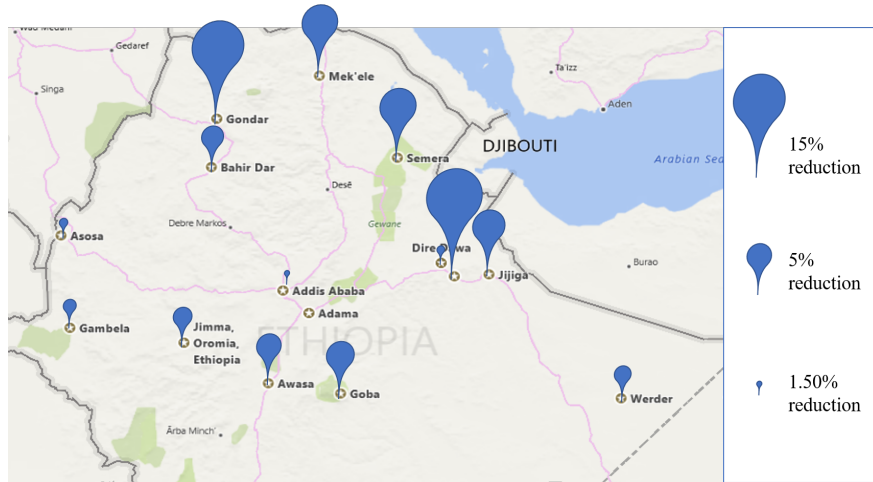


(b) Changes in teff transport patterns

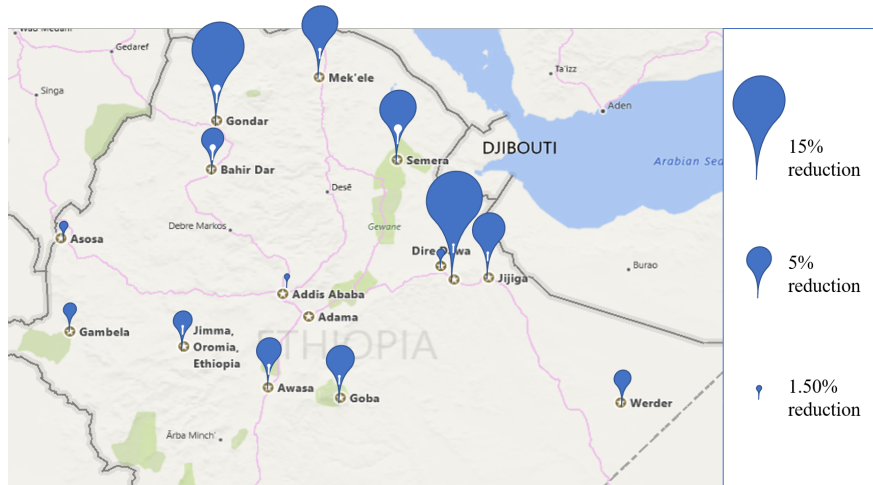
Figure 4.13: Crop producers' revenue and changes in teff transport patterns in different scenarios.

The details of the adaptation zone abbreviations are in [Figure 4.9](#).

pins are scaled to the percentage decrease in quantity sold in the markets). This includes markets in Gondar, Mekelle, Harar and Semera, which are all located in the Northern and Eastern part of



(a) Relative changes in quantities sold in domestic markets (in free export scenario) compared to complete export ban



(b) Relative changes in quantities sold in the domestic market (with 100 thousand quintals per season cap on export as well as free export)

Figure 4.14: Relative changes in quantities sold in domestic markets

the country. These disruptions are significantly smaller for a scenario of capped exports (white pins in Figure 4.14b). This result suggests that the government's proposal to allow regulated quantities of teff export [2] could achieve its desired goal of bringing in revenue while controlling shocks to local markets. However, this control comes at the cost of bringing in only about a tenth of the



Figure 4.15: Changes in consumer price under different scenarios

revenue obtainable in a more aggressive policy allowing exports.

Somali, the low fertility desert region of Ethiopia, is the region that is affected the most due to the teff export. It is costlier and unattractive for any producer to transport teff to Jijiga or Werder than to export. Thus, these locations suffer a price higher than the export price of teff. Further, this leads to decreased consumption of teff in these regions.

Consumer price of teff and revenue from teff export

Finally, we discuss the fluctuations in prices due to the shocks as one of the primary reasons to implement a ban on teff export. Currently, the domestic price of teff averages to about 74 United States dollars (USD) per quintal [133]. We note that under a free market for teff export, the price can rise to as high as 91 USD per quintal. This amounts to over 22% increase in the price of teff. While the inflation is not a huge number on a percentage basis, in the model it has the ability wipe out teff consumption in certain regions of the country (for example the moist lowlands (M1) adaptation zone), forcing the population in these regions to substitute other sources

of food, including wheat, barley and sorghum. Under no taxation for export, the export price is commensurate with the average domestic price. We also note that under the considered milder restrictions on export as opposed to a complete ban, namely a cap of 100,000 quintals or a million quintals of teff export per season, the inflation is relatively smaller at 0.58% and 7.5%. However, if there is an increase in the global demand for teff, even under these milder restrictions on export, domestic inflation can be as high as 17% and 18% respectively. In other words, we observe that the domestic prices of teff could be quite sensitive to global fluctuations of teff demand compared to governmental policies on teff export. The details of the observations are shown in [Figure 4.15](#).

4.3.5 Conclusion

In this work, we present the DECO2 model that can model shifts in equilibrium in Ethiopian food market by a detailed segregation of regions by their agroeconomic conditions for food production and consumption and a detailed market modeling for food transport and marketing at various parts of the country.

Compared to the partial equilibrium model presented in Bakker et al. [14], we have a more detailed modeling of food production using the adaptation zones which group regions with similar agro-economic properties together. We also have a more sophisticated modeling of markets across the country and transport between them, which helps in quantifying the distribution of welfare as well as domestic inflation under a range of government policies for teff export. The Thiessen polygons constructed with the markets as the seeds for the Voronoi tessellation, to mimic that the producers sell into the market of their general region.

We use the status of a complete ban on teff export as the base case. We then use the DECO2 model to identify shifts in equilibrium under scenarios including a completely free international

market for teff, and limited export quantities of 100,000 quintals per season and a million quintals per season, respectively. We also run these scenarios under higher global demand for teff. We observe that a lion's share of the additional international revenue is enjoyed by the distributors and storage operators as opposed to crop producers, unless they are large stakeholders themselves. Though there is a minor increase in national revenue during a typical year, a surge in the global demand for teff could cause significant harm to consumers across the country, especially to the ones in Northern and Eastern part of the country.

4.4 Conclusion and discussion

In this chapter, we showed how complementarity constraints can be used to model economic equilibrium problems to understand the North American natural gas market and the Ethiopian food market. While solving the former problem, we also developed an algorithm that approximates the covariance of the solution of a nonlinear complementarity problem under certain regularity conditions. We used this algorithm to also develop a sensitivity metric that can identify the parameters that contribute most to the variance in the solution of a stochastic complementarity problem. We then used these methods in the North American natural gas model, developed in Feijoo et al. [69], and identified the relevant parameters that the policymaker could alter in order to achieve a new market equilibrium.

Next, we apply complementarity constraints to model equilibrium in the Ethiopian food market. In particular, we analyze the current government policy to ban teff exports amidst increased domestic inflation due to global stimuli, against other scenarios like allowing a free export market or a market with caps on the maximum allowed quantity to export. We quantify the changes in domestic prices, regional differences in welfare due to such policy amidst differing global demand

for teff.

While we solve complementarity problems, it is common for researchers to be interested in the “best policy”, given that an equilibrium exists following the policy decision. Such problems can be posed as the so-called *Mathematical Program with Equilibrium Constraints* (MPECs). Ye and Zhu [142] provide a set of optimality conditions for MPECs, as the typical KKT conditions are no longer necessary, since standard constraint qualifications fail. Naturally, they require more sophisticated methods to solve. For example, Jara-Moroni, Pang, and Wächter [91] use a difference-of-convex optimization approach to solve linear MPECs to local optimum. Siddiqui and Gabriel [131] uses a SOS1-based approach to solve these problems with an energy application.

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Chapter 5

Mixed-integer bilevel programming

5.1 Introduction

In this chapter, we discuss mixed-integer bilevel linear programs (MIBLPs) of the form described in [Definition 12](#). We restate the definition below.

Definition 12 (Mixed-integer bilevel program). *Given $A \in \mathbb{Q}^{m_\ell \times n_\ell}$, $B \in \mathbb{Q}^{m_\ell \times n_f}$, $b \in \mathbb{Q}^{m_\ell}$, $f \in \mathbb{Q}^{n_f}$, $D \in \mathbb{Q}^{m_f \times n_f}$, $C \in \mathbb{Q}^{m_f \times n_\ell}$, $g \in \mathbb{Q}^{m_f}$, $\mathcal{I}_L \subseteq [n_\ell]$, $\mathcal{I}_F \subseteq [n_f]$, define*

$$S = S^1 \cap S^2 \cap S^3$$

$$S^1 = \{(x, y) \in \mathbb{R}^{n_\ell + n_f} : Ax + By \leq b\} \quad (2.1b)$$

$$S^2 = \left\{ (x, y) \in \mathbb{R}^{n_\ell + n_f} : y \in \arg \max_y \{f^T y : Dy \leq g - Cx, y_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_F\} \right\} \quad (2.1c)$$

$$S^3 = \{(x, y) \in \mathbb{R}^{n_\ell + n_f} : x_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_L\}$$

The mixed-integer bilevel program (MIBLP) is to find an element in the set S or to show that the set S is empty. We call the variables denoted by x as upper-level variables or leader's variables and we call the variables denoted by y as lower-level variables or follower's variables. We call the constraint defined by [eq. \(2.1c\)](#) as the bilevel constraint. If $\mathcal{I}_L = \mathcal{I}_F = \emptyset$, i.e., if all variables are

continuous, then we call the problem a continuous bilevel program (CBL). Further if $\mathcal{I}_F = \emptyset$, i.e., all follower's variables are continuous, then we call the problem a bilevel program with upper-level integer (BLP-UI).

The bilevel programming problem stated in [eq. \(2.1\)](#) has a long history, with traditions in theoretical economics (see, for instance, [\[114\]](#), which originally appeared in 1979) and operations research (see, for instance, [\[37, 94\]](#)). While much of the research community's attention has focused on the continuous case, there is a growing literature on bilevel programs with integer variables, starting with early work in the 1990s by Bard and Moore [\[17, 116\]](#) up until to a more recent surge of interest [\[53, 72, 73, 105, 106, 134, 138, 143\]](#). Research has largely focused on algorithmic concerns, with a recent focus on leveraging recent advancements in cutting plane technique. Typically, these algorithms restrict how integer variables appear in the problem. For instance, Wang and Xu [\[138\]](#) consider the setting where all variables are integer-valued whereas Fischetti et al. [\[73\]](#) allow continuous variables but no continuous variables of the leader in the follower's problem.

In this chapter, we give an algorithm that can solve a relaxation of MIBLPs. The algorithm runs in polynomial time, provided the number of follower variables and the number of integer variables of the leader is fixed.

In [Section 5.2](#), we discuss Stackelberg games, a problem that can be formulated and solved as a bilevel program. In [Section 5.3](#), we discuss some of the previously known results which are used in this chapter. In [Section 5.4](#), we present our key results which are proved in [Section 5.5](#). Finally we conclude the chapter in [Section 5.6](#).

5.2 Applications for mixed-integer bilevel programs

Bilevel optimization problems arise from a problem in economics known as the Stackelberg game. To motivate Stackelberg games, consider the following problem. Let us assume the electricity market in a country.

In a country A, let us say that the power-generating company has both coal-based power plants and solar-based power plants. The company has different costs of production for both these plants and has to serve for demand in the country. Let C_c be the unit cost of production from the coal-based plant and C_s be the unit cost of production from the solar plant. Let us say that the producer produces x_c and x_s quantity of electricity from each of these plants. However, the government restricts the maximum quantity the producer can produce from the coal plants, giving rise to constraints of the form $x_c \leq \bar{x}_c$ where \bar{x}_c is the limit imposed by the government. Let us also say, the power plants have to cater to a certain demand, $x_c + x_s \geq d - x_i$ where d is the demand at the location and x_i is the quantity the government decides to import. Let us also say, the producer switches on the solar generator only if it produces at least \underline{x}_s amount of electricity, or chooses to keep it off, as the fixed costs to operate the solar-based plant is high. This gives an integer constraint namely, $x_s \geq \underline{x}_s b_s$ where b_s is a binary variable that is 1 if the solar-based plant is on and 0 otherwise. Also, let \bar{x}_s be the capacity of the solar power plant. Now the power plant operator's problem is a mixed-integer linear program as follows:

$$\begin{array}{ll} \text{Minimize} & : \quad C_c x_c + C_s x_s \\ & x_c, x_s, b_s \end{array} \quad (5.2a)$$

such that

$$x_c, x_s \geq 0 \quad (5.2b)$$

$$x_c \leq \overline{x}_c \quad (5.2c)$$

$$x_c + x_s \geq d - x_i \quad (5.2d)$$

$$x_s \geq \underline{x}_s b_s \quad (5.2e)$$

$$x_s \leq \overline{x}_s b_s \quad (5.2f)$$

$$b_s \in \{0, 1\} \quad (5.2g)$$

Here the variables \overline{x}_c and x_i are *not* decided by the power-plant operator but by the government. The government might import electricity at a price C_i and perhaps coal emission has a cost of C_e per unit quantity. So the problem solved by the government is an MIBLP as follows:

$$\underset{x_i, \overline{x}_c}{\text{Minimize}} \quad : \quad C_e x_s + C_i x_i \quad (5.3a)$$

subject to

$$x_s \in \text{Solution of } \text{eq. (5.2)} \quad (5.3b)$$

This is a simple example of a mixed-integer bilevel problem in a practical application. We refer the readers to [73] for more applications where MIBLPs are used.

5.3 Mixed-integer bilevel programs and complexity

First we recall a part of [Theorem 14](#) and discuss its implications

Theorem 14 (Hardness of problems).

(v) [16] *Bilevel programming with $\mathcal{I}_L = \mathcal{I}_F = \emptyset$, i.e., continuous bilevel programming is NP-complete.*

(vi) [105] *Mixed-integer bilevel programming is Σ_2^P -complete.*

Statement (v) in Theorem 14 says that unless $P=NP$, which is conjectured to be extremely unlikely, there exists no algorithm that can solve the continuous bilevel program. Further, statement (vi) states that there does not exist an algorithm that can solve MIBLPs even in exponential time unless the polynomial hierarchy collapses to the second level, which is also conjectured to be very unlikely. In other words, every algorithm to solve a general MIBLP uses at least $\Omega(2^{2^n})$ elementary operations.

For the purpose of this chapter we define $\text{poly}(a, b, c, \dots)$ to define a polynomial in its arguments a, b, c, \dots and $\vartheta(a, b, c, \dots)$ to be an arbitrary function. Further, we use ϕ to denote the number of bits required to store the largest entry in the data of the problem, where the data could be matrices or vectors or scalars used to describe the problem.

Amidst the strong negative results on finding algorithms to solve MIBLPs, Köppe, Queyranne, and Ryan [98] give an algorithm to solve an MIBLP in *polynomial time* under specific conditions. We state the theorem formally below.

Theorem 69 (Theorem 3.1 in Köppe et al. [98]). *Let n_f be a fixed number and $\mathcal{I}_F = [n_f]$. Also let $\mathcal{I}_L = \emptyset$. This subclass of MIBLPs can be solved in $O(\text{poly}(m_\ell, m_f, n_\ell, \phi))$ time.*

In other words, the authors in Köppe et al. [98] state that if there are fixed number of follower variables and the follower's problem is a pure integer program while the leader's variables are all continuous, then the problem can be solved in polynomial time.

The algorithm that Köppe et al. [98] use in this work hinges on the central result of Eisenbrand and Shmonin [64]. We extend the definition of mixed-integer representable sets below before stating Theorem 71 due to Eisenbrand and Shmonin [64].

Definition 70. Let $S \subseteq \mathbb{R}^m$ be a mixed-integer representable set (refer [Definitions 8](#) and [10](#)). We say S is a type 1 set of order p if there exists a rational polyhedron $Q \subseteq \mathbb{R}^{m+p}$ such that

$$S := Q/\mathbb{Z}^p := \{\sigma \in \mathbb{R}^m : \exists z \in \mathbb{Z}^p \text{ such that } (\sigma, z) \in Q\} \quad (5.4)$$

Further, we say S is a type 2 set of order (p, q) if there exists a rational polyhedron $Q \subseteq \mathbb{R}^{m+p+q}$ such that

$$S := Q/(\mathbb{Z}^p \times \mathbb{R}^q) := \{\sigma \in \mathbb{R}^m : \exists z \in \mathbb{Z}^p \text{ and } y \in \mathbb{R}^q \text{ such that } (\sigma, z, y) \in Q\} \quad (5.5)$$

Observe that a type 1 set of order p is a type 2 set of order $(p, 0)$.

The above definition quantifies the number of integer variables and continuous variables “projected out” of a mixed-integer set to get the mixed-integer representable set S . This is necessary to state the complexity results of Eisenbrand and Shmonin [\[64\]](#).

Theorem 71 ([\[64\]](#)). Let $S \subseteq \mathbb{R}^m$ be a type 1 set of order p . Let $M \in \mathbb{Q}^{m \times n}$. The sentence

$$\forall \sigma \in S \quad \exists x \in P_\sigma \cap \mathbb{Z}^n \quad \text{where } P_\sigma := \{x \in \mathbb{R}^n : Mx \leq \sigma\} \quad (5.6)$$

can be decided in polynomial time if p and n are fixed. In other words, the decision problem can be solved in $O(\text{poly}(m, \phi)\vartheta(n, p))$ time.

In other words, if the RHS σ of the polyhedron changes over a mixed-integer representable set S , we can answer if all the pure-integer programs hence obtained are feasible in polynomial time, provided the number of variables in the pure-integer program as well as the number of variables projected out to obtain S are fixed, provided all the variables that are projected out are integer variables.

In our work, we extend the results to more general settings and apply them to obtain algorithms to solve MIBLPs in some settings and a relaxation of MIBLPs in others. We compile our results as a part of this work in [Section 5.4](#).

5.4 Key results

First, we extend [Theorem 71](#) to a setting where S can be a type 2 set.

Theorem 72. *Let $S \subseteq \mathbb{R}^m$ be a type 2 set of order (p, q) . Let $M \in \mathbb{Q}^{m \times n}$ be a rational matrix.*

Then the sentence

$$\forall \sigma \in S \quad \exists x \in P_\sigma \cap \mathbb{Z}^n \quad \text{where } P_\sigma := \{x \in \mathbb{R}^n : Mx \leq \sigma\} \quad (5.7)$$

can be decided in time bounded by $O(\text{poly}(m, q, \phi) \vartheta(n, p))$. i.e., if n and p are fixed, the problem can be solved in polynomial time.

Note that in this theorem, we do not require q to be fixed. The algorithm we provide has a complexity that is polynomial in q . Clearly [Theorem 72](#) implies [Theorem 71](#).

While [Theorems 71](#) and [72](#) deal with existence of pure-integer points in the family of polyhedra P_σ , the theorem below extends it for mixed-integer points in all members of the family. Also, though [Theorem 73](#) implies [Theorem 72](#), we state both the theorem separately, as we use [Theorem 72](#) to prove [Theorem 73](#).

Theorem 73. *Let $S \subseteq \mathbb{R}^m$ be a type 2 set of order (p, q) . Let $M \in \mathbb{Q}^{m \times (n_1 + n_2)}$ be a rational matrix. Then the sentence*

$$\forall \sigma \in S \quad \exists x \in P_\sigma \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \quad \text{where } P_\sigma := \left\{x \in \mathbb{R}^{(n_1 + n_2)} : Mx \leq \sigma\right\} \quad (5.8)$$

can be decided in time bounded by $O(\text{poly}(m, q, \phi) \vartheta(p, n_1, n_2))$. In other words, the problem can be decided in polynomial time if p , n_1 and n_2 are fixed.

Now we define a relaxation of the mixed-integer bilevel program defined in [Definition 12](#). We then show that under special assumptions, this relaxation can be solved in polynomial time.

Definition 74 (ε -mixed-integer bilevel program). *Given $A \in \mathbb{Q}^{m_\ell \times n_\ell}$, $B \in \mathbb{Q}^{m_\ell \times n_f}$, $b \in \mathbb{Q}^{m_\ell}$, $f \in \mathbb{Q}^{n_f}$, $D \in \mathbb{Q}^{m_f \times n_f}$, $C \in \mathbb{Q}^{m_f \times n_\ell}$, $g \in \mathbb{Q}^{m_f}$, $\mathcal{I}_L \subseteq [n_\ell]$, $\mathcal{I}_F \subseteq [n_f]$, define for $\varepsilon > 0$*

$$\begin{aligned} S_\varepsilon &= S^1 \cap S^2(\varepsilon) \cap S^3 \quad \text{where} \\ S^2(\varepsilon) &= \left\{ (x, y) \in \mathbb{R}^{n_\ell + n_f} : f^T y + \varepsilon \geq \max_y \{ f^T y : Dy \leq g - Cx, y_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}_F \} \right\} \end{aligned} \quad (5.9a)$$

where S_1 and S_3 are as defined in [Definition 12](#). Then the ε -mixed-integer bilevel program (ε -MIBLP) is to either find an element in the set S or to show that $S = \emptyset$.

Note that, the bilevel constraint [eq. \(2.1c\)](#) is relaxed in [Definition 74](#) such that the follower's problem is not solved to optimality but ensured that the follower is not more than ε away from her optimal decision.

Theorem 75. *For $0 < \varepsilon < 1$, ε -MIBLP can be solved in time bounded by $O(\text{poly}(m_\ell, n_\ell, m_f, \log(1/\varepsilon), \phi) \vartheta(|\mathcal{I}_L|, n_f))$. In other words, if $|\mathcal{I}_L|$, $|\mathcal{I}_F|$ and n_f are fixed, then the ε -MIBLP can be solved in polynomial time.*

Note that ε is part of the data for the problem. So strictly speaking, the dependence on the size of ε is already contained in ϕ , however, we state the polynomial dependence on the size of ε explicitly to emphasize its significance.

Corollary 76. *The following are true. Unless $P = NP$, there exists no polynomial time that can solve an MIBLP under any of the following conditions:*

- (a) *Fixing n_f and hence $|\mathcal{I}_F|$ while allowing n_ℓ and $|\mathcal{I}_L|$ to vary.*
- (b) *Fixing n_ℓ and hence $|\mathcal{I}_L|$ while allowing n_f and $|\mathcal{I}_F|$ to vary.*
- (c) *Fixing $|\mathcal{I}_L|$ and $|\mathcal{I}_F|$ while allowing n_ℓ and n_f to vary.*

These theorems are proved formally in [Section 5.5](#).

5.5 Proofs of Theorems 72, 73 and 75

First we prove the following lemma which is required in the proof of Theorem 72.

Lemma 77. *Let $Q \subseteq \mathbb{R}^{m+p+q}$ be a rational polyhedron. Let $M \in \mathbb{Q}^{m \times n}$ be given. We claim that the following are equivalent.*

1. $\forall \sigma \in Q/(\mathbb{Z}^p \times \mathbb{R}^q), \exists x \in P_\sigma \cap \mathbb{Z}^n$ where $P_\sigma := \{x \in \mathbb{R}^n : Mx \leq \sigma\}$.

2. $\forall \tilde{\sigma} \in Q/\mathbb{Z}^p, \exists \tilde{x} \in \widetilde{P}_{\tilde{\sigma}} \cap \mathbb{Z}^{n+1}$ where

$$\widetilde{P}_{\tilde{\sigma}} := \{\tilde{x} = (x, x') \in \mathbb{R}^n \times \mathbb{R} : Mx \leq \text{Proj}_{\mathbb{R}^m}(\tilde{\sigma}), x' \leq \tilde{\sigma}_{m+1}, \dots, x' \leq \tilde{\sigma}_{m+q}\} \quad (5.10)$$

Lemma 77 establishes the equivalence solving the so-called parametric integer program in Theorem 71 with one additional variable and solving a similar problem where the mixed-integer representable set over which σ can vary is extended to be of type 2 as well. We prove the lemma below.

Proof. First we prove that 2 \implies 1. Let us assume that statement 2 holds. Now, consider some $\sigma \in Q/(\mathbb{Z}^p \times \mathbb{R}^q)$. By definition, this means $\exists \tilde{\sigma} \in Q/\mathbb{Z}^p \subseteq \mathbb{R}^{m+q}$ such that $\sigma = \text{Proj}_{\mathbb{R}^m}(\tilde{\sigma})$. From statement 2, we know there exists $\tilde{x} = (x, x')$ such that $Mx \leq \sigma$ and $x \in \mathbb{Z}^n$. But $x \in P_\sigma \cap \mathbb{Z}^n$, thus proving 1.

Conversely, consider some $\tilde{\sigma} \in Q/\mathbb{Z}^p$. Let $\sigma = \text{Proj}_{\mathbb{R}^m}(\tilde{\sigma})$. By statement 1, $\exists x \in \mathbb{Z}^n$ such that $Mx \leq \sigma$. Choose $x' = \lfloor \min_i \tilde{\sigma}_i \rfloor$, i.e., the floor of the smallest coordinate of $\tilde{\sigma}$. Now $(x, x') \in \widetilde{P}_{\tilde{\sigma}} \cap \mathbb{Z}^{n+1}$. ■

Now, using Lemma 77, we prove Theorem 72.

Proof of Theorem 72. Since S is a type 2 set, we can write $S = Q/(\mathbb{Z}^p \times \mathbb{R}^q)$ for some rational polyhedron $Q \in \mathbb{R}^{m+p+q}$. Thus sentence in eq. (5.7) is in the form of statement 1 in Lemma 77.

But by [Lemma 77](#), this is equivalent to checking the statement 2. However, statement 2 corresponds to a similar problem with just one additional variable but $\tilde{\sigma}$ varying over a type 1 set Q/\mathbb{Z}^p . Here if p and n are fixed, then the size of description of $\tilde{P}_{\tilde{\sigma}}$ is polynomial in the size of description of P_{σ} and q . But this sentence can be checked in polynomial time using [Theorem 71](#). \blacksquare

Now that we allow σ in P_{σ} to vary over a type 2 mixed-integer representable set without any loss in computational efficiency, we prove [Theorem 73](#) which is an extension of [Theorem 71](#) for mixed-integer points in a parametric polyhedron.

Proof of Theorem 73. Let $Q \subseteq \mathbb{R}^{m+p+q}$ be the rational polyhedron such that $S = Q/(\mathbb{Z}^p \times \mathbb{R}^q)$.

Now consider $P'_{\sigma} = \text{Proj}_{\mathbb{R}^{n_1}}(P_{\sigma})$. Clearly $P_{\sigma} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \neq \emptyset \iff P'_{\sigma} \cap \mathbb{Z}^{n_1} \neq \emptyset$.

By Fourier-Motzkin procedure, there exists a matrix $U \in \mathbb{Q}^{r \times m}$ such that $P'_{\sigma} = \{x \in \mathbb{R}^{n_1} : UMx \leq U\sigma\}$ with $r \leq m^{2^{n_2}}$ where U is dependent *only* on M [[126](#)]. Note that r is polynomially bounded (in m) because we assume n_2 is a fixed constant.

Thus, it is equivalent to check the statement

$$\forall \sigma' \in \{U\sigma : \sigma \in S\} \quad \exists x \in P'_{\sigma'} \cap \mathbb{Z}^{n_1} \quad \text{where } P'_{\sigma'} := \{x : M'x \leq \sigma'\} \quad (5.11)$$

with $M' = UM$.

We now observe that if Q has an inequality description $Q = \{(\sigma, z, y) \in \mathbb{R}^m \times \mathbb{Z}^p \times \mathbb{R}^q : B\sigma + Cz + Ey \leq d\}$ for some matrices B, C, E of appropriate dimensions, and right hand side d of appropriate dimension, then

$$\{U\sigma : \sigma \in S\} = \left\{ \begin{array}{l} \exists z \in \mathbb{Z}^p, y \in \mathbb{R}^q, \sigma \in \mathbb{R}^m, \text{ such that} \\ \sigma' \in \mathbb{R}^r : B\sigma + Cz + Ey \leq d \\ \sigma' = U\sigma \end{array} \right\} \quad (5.12)$$

The statement in [eq. \(5.11\)](#) is then equivalent to checking

$$\forall \sigma' \in Q' / (\mathbb{Z}^p \times \mathbb{R}^q \times \mathbb{R}^m), \exists x \in P'_{\sigma'} \cap \mathbb{Z}^{n_1}, \quad (5.13)$$

where $Q' := \left\{ (\sigma', z, y, \sigma) \in \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m : \begin{array}{l} B\sigma + Cz + Ey \leq d \\ \sigma' = U\sigma \end{array} \right\}$ is a rational polyhedron

whose description size is polynomial in the description size of Q (note that we use the fact the r is polynomially bounded in m). Since p and n_1 are fixed constants, statement [eq. \(5.13\)](#) can be checked in polynomial time by [Theorem 72](#). ■

Corollary 78. *Given a type 2 set S of fixed order (p, q) and fixed n_1 and n_2 , with $M \in \mathbb{Q}^{m \times (n_1 + n_2)}$, the sentence*

$$\exists \sigma \in S \quad : \quad \{x : Mx \leq \sigma\} \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) = \emptyset \quad (5.14)$$

can be decided in polynomial time.

Proof. This follows from negating the sentence in [Theorem 73](#). ■

With the theorems that extend [Theorem 71](#) from Eisenbrand and Shmonin [64] proved, we now prove the central result [Theorem 75](#) of this chapter that a relaxation of MIBLP can be solved in polynomial time.

Proof of Theorem 75. Without loss of generality, let us write that the first $|\mathcal{I}_F|$ of the follower's variables are constrained to be integers. Also note that when n_ℓ is fixed, $|\mathcal{I}_L| \leq n_\ell$ is also fixed. Similarly when n_f is fixed, so is $|\mathcal{I}_F| \leq n_f$.

Now note that for $\varepsilon > 0$, $(x, y) \in S_2(\varepsilon)$ if

$$\left\{ y' : \begin{array}{l} f^T y' - \varepsilon \geq f^T y \\ Dy' \leq g - Cx \end{array} \right\} \cap \left(\mathbb{Z}^{|\mathcal{I}_F|} \times \mathbb{R}^{n_f - |\mathcal{I}_F|} \right) = \emptyset \quad (5.15)$$

$$\{y' : My' \leq \sigma\} \cap \left(\mathbb{Z}^{|\mathcal{I}_F|} \times \mathbb{R}^{n_f - |\mathcal{I}_F|} \right) = \emptyset \quad (5.16)$$

where $M = \begin{pmatrix} f^T \\ D \end{pmatrix}$ and $\sigma = \begin{pmatrix} f^T y + \varepsilon \\ g - Cx \end{pmatrix}$. The condition in the first row of the first set in [eq. \(5.15\)](#) looks for y' which is more than “ ε better” than y , while the second row exists that such a y' satisfies the inequality constraints that y should follow. The second set ensures that such a y' also satisfies the required integrality constraints. Note that M does not depend upon the point (x, y) whose feasibility is checked.

Now, (x, y) cannot be any point, but to be chosen from $S^1 \cap S^3$ (from the notation of [Definition 74](#)) along with it satisfying [eq. \(5.16\)](#), so that it is feasible to the ε -MIBLP. So the question transforms into the form

$$\exists \sigma \in T \quad : \quad \{x : Mx \leq \sigma\} \cap \left(\mathbb{Z}^{|\mathcal{I}_F|} \times \mathbb{R}^{n_f - |\mathcal{I}_F|} \right) = \emptyset \quad (5.17)$$

where

$$T = \left\{ \sigma : \sigma = \begin{pmatrix} f^T y + \varepsilon \\ g - Cx \end{pmatrix} \text{ for } (x, y) \in S^1 \cap S^3 \right\}$$

But from [Definition 74](#), $S^1 \cap S^3$ is a mixed-integer representable sets whose order is fixed if $|\mathcal{I}_L|$, $|\mathcal{I}_F|$ and n_f are fixed. Thus T , which is a linear transform of $S^1 \cap S^3$ is also a mixed-integer representable set of fixed order. Now the result follows from [Corollary 78](#). ■

Proof of [Corollary 76](#). (a) Observe that fixing $n_f = 0$ gives a mixed-integer linear program that is NP complete.

(b) Observe that fixing $n_\ell = 0$ gives a mixed-integer linear program that is NP complete.

(c) Bard [\[16\]](#) shows the NP completeness of a continuous bilevel program which has $|\mathcal{I}_F| = |\mathcal{I}_L| = 0$. ■

5.6 Conclusion

In this chapter, we defined and motivated the importance of mixed-integer bilevel optimization problems. Then, we extended the results from Eisenbrand and Shmonin [64] for more general settings. Using the extended version of these results, we motivate an ε -relaxation of the mixed-integer bilevel problem and show that if certain parameters are fixed, this problem can be solved in polynomial time. Then as a corollary, we deduce that no polynomial time algorithm can be found for certain versions of the problem.

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Chapter 6

Linear complementarity and mixed-integer bilevel representability

6.1 Introduction

Separate from the design of algorithms and questions of computational complexity, studying representability quantifies the family of sets which can be modeled using a class of optimization problems. The concept of representability was motivated in [Example 9](#) and formally defined in [Definition 8](#). Representability quantifies the types of sets over which one could optimize using a given class of optimization problems.

The classical paper of Jeroslow and Lowe [\[92\]](#) provides a characterization of sets that can be represented by mixed-integer linear feasible regions. They show that a set is the projection of the feasible region of a mixed-integer linear program (termed *MILP-representable*) if and only if it is the Minkowski sum of a finite union of polytopes and a finitely-generated integer monoid (concepts more carefully defined below). This result answers a long-standing question on the limits

of mixed-integer programming as a modeling framework. Jeroslow and Lowe’s result also serves as inspiration for recent interest in the representability of a variety of problems. See Vielma [136] for a review of the literature until 2015 and Basu et al. [21], Del Pia and Poskin [49, 50], Lubin et al. [107, 108, 109] for examples of more recent work.

To our knowledge, questions of representability have not even been explicitly asked of *continuous bilevel linear* (CBL) programs where $\mathcal{I}_L = \mathcal{I}_F = \emptyset$ in Definition 12. Accordingly, our initial focus concerns the characterizations of CBL-representable sets. In the first key result of our work (Theorem 94), we show that every CBL-representable set can also be modeled as the feasible region of a linear complementarity (LC) problem (in the sense of Cottle et al. [47]). Indeed, we show that both CBL-representable sets and LC-representable sets are precisely finite unions of polyhedra. Our proof method works through a connection to superlevel sets of piecewise linear convex functions (what we term *polyhedral reverse-convex sets*) that alternately characterize finite unions of polyhedra. In other words, an arbitrary finite union of polyhedra can be modeled as a continuous bilevel program, a linear complementarity problem, or an optimization problem over a polyhedral reverse-convex set.

A natural question arises: how can one relate CBL-representability and MILP-representability? Despite some connections between CBL programs and MILPs (see, for instance, Audet et al. [10]), the collection of sets they represent are incomparable (see Corollary 95 below). The Jeroslow-Lowe characterization of MILP-representability as the finite union of polytopes summed with a finitely-generated monoid has a fundamentally different geometry than CBL-representability as a finite union of polyhedra. It is thus natural to conjecture that MIBL-representability should involve some combination of the two geometries. We will see that this intuition is *roughly* correct, with an important caveat.

A distressing fact about MIBL programs, noticed early on in Moore and Bard [116], is that the feasible region of a MIBL program may not be *topologically closed* (maybe the simplest example illustrating this fact is Example 1.1 of Köppe et al. [98]). This throws a wrench in the classical narrative of representability that has largely focused on closed sets. Indeed, the recent work of Lubin et al. [108] is careful to study representability by *closed* convex sets. This focus is entirely justified. Closed sets are indeed of most interest to the working optimizer and modeler, since sets that are not closed may fail to have desirable optimality properties (such as the non-existence of optimal solutions). Accordingly, we aim our investigation on *closures* of MIBL-representable sets. In fact, we provide a complete characterization of these sets as unions of finitely many MILP-representable sets (Theorem 98). This is our second key result on MIBL-representability. The result conforms to the rough intuition of the last paragraph. MIBL-representable sets are indeed finite unions of other objects, but instead of these objects being polyhedra as in the case of CBL-programs, we now take unions of MILP-representable sets, reflecting the inherent integrality of MIBL programs.

To prove this second key result on MIBL-representability we develop a generalization of Jeroslow and Lowe’s theory to mixed-integer sets in *generalized polyhedra*, which are finite intersections of closed *and* open halfspaces. Indeed, it is the non-closed nature of generalized polyhedra that allows us to study the non-closed feasible regions of MIBL-programs. Specifically, these tools arise when we take the value function approach to bilevel programming, as previously studied in [53, 106, 135, 142]. Here, we leverage the characterization of Blair [32] of the value function of the mixed-integer program in the lower-level problem. Blair’s characterization leads us to analyze superlevel and sublevel sets of *Chvátal functions*. A Chvátal function is (roughly speaking) a linear function with integer rounding (a more formal definition later). Basu et al. [21] show that superlevel sets of Chvátal functions are MILP-representable. Sublevel sets are trickier, but for a familiar reason

— they are, in general, not closed. This is not an accident. The non-closed nature of mixed-integer bilevel sets, generalized polyhedra, and sublevel sets of Chvátal functions are all tied together in a key technical result that shows that sublevel sets of Chvátal functions are precisely finite unions of generalized mixed-integer linear representable (GMILP-representable) sets ([Theorem 100](#)). This result is the key to establishing our second main result on MIBL-representability.

In fact, showing that the sublevel set of a Chvátal function is the finite union of GMILP-representable sets is a corollary of a more general result. Namely, we show that the collection of sets that are finite unions of GMILP-representable sets forms an algebra (closed under unions, intersections, and complements). We believe this result is of independent interest.

The representability results in the mixed-integer case require rationality assumptions on the data. This is an inevitable consequence when dealing with mixed-integer sets. For example, even the classical result of Jeroslow and Lowe [\[92\]](#) requires rationality assumptions on the data. Without this assumption, the result does not even hold. The key issue is that the convex hull of mixed-integer sets are not necessarily polyhedral unless certain other assumptions are made, amongst which the rationality assumption is most common (see Dey and Morán [\[55\]](#) for a discussion of these issues).

Of course, it is natural to ask if this understanding of MIBL-representability has implications for questions of computational complexity. The interplay between representability and complexity is subtle. We show (in [Theorem 98](#)) that allowing integer variables in the leader’s decision x captures everything in terms of representability as when allowing integer variables in both the leader and follower’s decision (up to taking closures). However, we show (in [Theorem 101](#)) that the former is in \mathcal{NP} while the latter is Σ_p^2 -complete [\[105\]](#). This underscores a crucial difference between an integer variable in the upper level versus an integer variable in the lower-level, from a computational

complexity standpoint.

In summary, we make the following contributions. We provide geometric characterizations of CBL-representability and MIBL-representability (where the latter is up to closures) in terms of finite unions of polyhedra and finite unions of MILP-representable sets, respectively. In the process of establishing these main results, we also develop a theory of representability of mixed-integer sets in generalized polyhedra and show that finite unions of GMILP-representable sets form an algebra. This last result has the implication that finite unions of MILP-representable sets also form an algebra, up to closures.

The rest of the chapter is organized as follows. The main definitions needed to state our main results are found in [Section 6.2](#), followed in [Section 6.3](#) by self-contained statements of these main results. [Section 6.4](#) contains our analysis of continuous bilevel sets and their representability. [Section 6.5](#) explores representability in the mixed-integer setting. [Section 6.6](#) concludes.

6.2 Key definitions

This section provides the definitions needed to understand the statements of our main results collected in [Section 6.3](#). Concepts that appear only in the proofs of these results are defined later as needed. We begin with formal definitions of the types of sets we study in this chapter.

First we extend [Definition 11](#) in this chapter to include linear inequalities as follows.

Definition 79 (Linear complementarity sets). *A set $S \subseteq \mathbb{R}^n$ is a linear complementarity (LC) set, if there exist $M \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ and A, b of appropriate dimensions such that*

$$S = \left\{ x \in \mathbb{R}^n : x \geq 0, Mx + q \geq 0, x^\top (Mx + q) = 0, Ax \leq b \right\}.$$

Sometimes, we represent this using the alternative notation

$$0 \leq x \perp Mx + q \geq 0$$

$$Ax \leq b.$$

A linear complementarity set will be labeled rational if all the entries in A, M, b, q are rational.

As an example, the set of all n -dimensional binary vectors is a linear complementarity set.

Indeed, they can be modeled as $0 \leq x_i \perp (1 - x_i) \geq 0$ for $i \in [n]$.

Definition 80 (Polyhedral convex function). A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a polyhedral convex function with k pieces if there exist $\alpha^1, \dots, \alpha^k \in \mathbb{R}^n$ and $\beta_1, \dots, \beta_k \in \mathbb{R}$ such that

$$f(x) = \max_{j=1}^k \{ \langle \alpha^j, x \rangle - \beta_j \}.$$

A polyhedral convex function will be labeled rational if all the entries in the affine functions are rational.

Note that f is a maximum of finitely many affine functions. Hence f is always a convex function.

Definition 81 (Polyhedral reverse-convex set). A set $S \in \mathbb{R}^n$ is a polyhedral reverse-convex (PRC) set if there exist $n' \in \mathbb{Z}_+$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and polyhedral convex functions f_i for $i \in [n']$ such that

$$S = \{x \in \mathbb{R}^n : Ax \leq b, f_i(x) \geq 0 \text{ for } i \in [n']\}.$$

A polyhedral reverse-convex set will be labeled rational if all the entries in A, b are rational, and the polyhedral convex functions f_i are all rational.

One of the distinguishing features of MIBL sets is their potential for not being closed. This was discussed in the introduction and a concrete example is provided in [Lemma 130](#). To explore the possibility of non-closedness we introduce the notion of generalized polyhedra.

Definition 82 (Generalized, regular, and relatively open polyhedra). *A generalized polyhedron is a finite intersection of open and closed halfspaces. A bounded generalized polyhedron is called a generalized polytope. The finite intersection of closed halfspaces is called a regular polyhedron. A bounded regular polyhedron is a regular polytope. A relatively open polyhedron P , is a generalized polyhedron such that $P = \text{relint}(P)$. If relatively open polyhedron P is bounded, we call it a relatively open polytope.*

Such sets will be labeled rational if all the defining halfspaces (open or closed) can be given using affine functions with rational data.

Note that the closure of a generalized or relatively open polyhedron is a regular polyhedron. Also, singletons are, by definition, relatively open polyhedra.

Definition 83 (Generalized, regular and relatively open mixed-integer sets). *A generalized (respectively, regular and relatively open) mixed-integer set is the set of mixed-integer points in a generalized (respectively, regular and relatively open) polyhedron.*

Such sets will be labeled rational if the corresponding generalized polyhedra are rational.

Our main focus is to explore how collections of the above objects can be characterized and are related to one another. To facilitate this investigation we employ the following notation and vocabulary. Let \mathcal{T} be a family of sets. These families will include objects of potentially different dimensions. For instance, the family of polyhedra will include polyhedra in \mathbb{R}^2 as well as those in \mathbb{R}^3 . We will often not make explicit reference to the ambient dimension of a member of the family \mathcal{T} , especially when it is clear from context unless explicitly needed. For a family \mathcal{T} , the subfamily of bounded sets in \mathcal{T} will be denoted by $\overline{\mathcal{T}}$. Also, $\text{cl}(\mathcal{T})$ is the family of the closures of all sets in \mathcal{T} . When referring to the *rational* members of a family (as per definitions above), we will use the notation $\mathcal{T}(\mathbb{Q})$.

We are not only interested in the above sets, but also linear transformations of these sets. This notion is captured by the concept of representability. While we defined it before in [Definition 8](#), here we extend the definition for rationally representable sets too.

Definition 84 (Representability). *Given a family of sets \mathcal{T} , S is called a \mathcal{T} -representable set or representable by \mathcal{T} if there exists a $T \in \mathcal{T}$ and a linear transform L such that $S = L(T)$. The collection of all such \mathcal{T} -representable sets is denoted \mathcal{T}_R . We use the notation $\mathcal{T}_R(\mathbb{Q})$ to denote the images of the rational sets in \mathcal{T} under rational linear transforms, i.e., those linear transforms that can be represented using rational matrices.*¹

Remark 85. The standard definition of representability in the optimization literature uses projections as opposed to general linear transforms. However, under mild assumption on the family \mathcal{T} , it can be shown that \mathcal{T}_R is simply the collection of sets that are *projections* of sets in \mathcal{T} . Since projections are linear transforms, we certainly get all projections in \mathcal{T}_R . Now consider a set $S \in \mathcal{T}_R$, i.e., there exists a set $T \in \mathcal{T}$ and a linear transform L such that $S = L(T)$. Observe that $S = \text{Proj}_x\{(x, y) : x = L(y), y \in T\}$. Thus, if \mathcal{T} is a family that is closed under the addition of affine subspaces (like $x = L(y)$ above), and addition of free variables (like the set $\{(x, y) : y \in T\}$), then \mathcal{T}_R does not contain anything beyond projections of sets in \mathcal{T} . All families considered in this chapter are easily verified to satisfy these conditions.

One can immediately observe that $\mathcal{T} \subseteq \mathcal{T}_R$ since the linear transform can be chosen as the identity transform. However, the inclusion may or may not be strict. For example, it is well known that if \mathcal{T} is the set of all polyhedra, then $\mathcal{T}_R = \mathcal{T}$. However, if \mathcal{T} is the family of all (regular) mixed-integer sets then $\mathcal{T} \subsetneq \mathcal{T}_R$.

¹We will never need to refer to general linear transforms of rational sets, or rational linear transforms of general sets in a family \mathcal{T} ; so we do not introduce any notation for these contingencies.

When referencing specific families of sets we use the following notation. The family of all linear complementarity sets is \mathcal{T}^{LC} , continuous bilevel sets is \mathcal{T}^{CBL} and the family of all polyhedral reverse-convex sets is \mathcal{T}^{PRC} . The family of mixed-integer sets is \mathcal{T}^{MI} . The family of MIBL sets is \mathcal{T}^{MIBL} . The family of BLP-UI sets is \mathcal{T}^{BLP-UI} . We use \mathcal{P} to denote the family of *finite* unions of polyhedra and \mathcal{T}^{D-MI} to denote the family of finite unions of sets in \mathcal{T}^{MI} . We use $\widehat{\mathcal{T}^{MI}}$ to denote the family of generalized mixed-integer sets and $\widehat{\mathcal{T}^{D-MI}}$ to denote the family of sets that can be written as finite unions of sets in $\widehat{\mathcal{T}^{MI}}$. The family of all integer cones is denoted by \mathcal{T}^{IC} . This notation is summarized in [Table 6.1](#).

We make a useful observation at this point.

Lemma 86. *The family \mathcal{T}_R^{D-MI} is exactly the family of finite unions of MILP-representable sets, i.e., finite unions of sets in \mathcal{T}_R^{MI} . Similarly, $\widehat{\mathcal{T}_R^{D-MI}}$ is exactly the family of finite unions of sets in $\widehat{\mathcal{T}_R^{MI}}$. The statements also holds for the rational elements, i.e., when we consider $\mathcal{T}_R^{D-MI}(\mathbb{Q})$, $\mathcal{T}_R^{MI}(\mathbb{Q})$, $\widehat{\mathcal{T}_R^{D-MI}}(\mathbb{Q})$ and $\widehat{\mathcal{T}_R^{MI}}(\mathbb{Q})$, respectively.*

Proof. Consider any $T \in \mathcal{T}_R^{D-MI}$. By definition, there exist sets $T_1, \dots, T_k \in \mathcal{T}^{MI}$ and a linear transformation L such that $T = L\left(\bigcup_{i=1}^k T_i\right) = \bigcup_{i=1}^k L(T_i)$ and the result follows from the definition of MILP-representable sets.

Now consider $T \subseteq \mathbb{R}^n$ such that $T = \bigcup_{i=1}^k T_i$ where $T_i \in \mathcal{T}_R^{MI}$ for $i \in [k]$. By definition of $T_i \in \mathcal{T}_R^{MI}$, we can write $\mathbb{R}^n \supseteq T_i = \{L_i(z^i) : z^i \in T'_i\}$ where $T'_i \subseteq \mathbb{R}^{n_i} \times \mathbb{R}^{n'_i}$ is a set of mixed-integer points in a polyhedron and $L_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n'_i} \mapsto \mathbb{R}^n$ is a linear transform. In other words, $T'_i = \{z^i = (x^i, y^i) \in \mathbb{R}^{n_i} \times \mathbb{Z}^{n'_i} : A^i x^i + B^i y^i \leq b^i\}$. Let us define

$$\tilde{T}_i = \{(x, x^1, \dots, x^k, y^1, \dots, y^k) : \forall j \in [k], x^j \in \mathbb{R}^{n_j}, y^j \in \mathbb{R}^{n'_j}; x = L_i(x^i), A^i x^i + B^i y^i \leq b^i\}. \quad (6.1)$$

Clearly, for all $i \in [k]$, $\tilde{T}_i \subseteq \mathbb{R}^{n+\sum_{j=1}^k (n_j+n'_j)}$ is a polyhedron and projecting $\tilde{T}_i \cap$

$(\mathbb{R}^{n+\sum_{j=1}^k n_j} \times \mathbb{Z}^{\sum_{j=1}^k n'_j})$ over the first n variables gives T_i . Let us denote the projection from $(x, x^1, \dots, x^k, y^1, \dots, y^k)$ onto the first n variables by L . Since any linear operator commutes with finite unions, we can write,

$$T = \left\{ L(z) : z = (x, x^1, \dots, x^k, y^1, \dots, y^k) \in \left(\bigcup_{i=1}^k \tilde{T}_i \right) \cap (\mathbb{R}^{n'} \times \mathbb{Z}^{n'') \right\},$$

where $n' = n + \sum_{j=1}^k n_j$ and $n'' = \sum_{j=1}^k n'_j$, proving the first part of the lemma.

The second part about $\widehat{\mathcal{T}_R^{D-MI}}$ follows along very similar lines and is not repeated here. The rational version also follows easily by observing that nothing changes in the above proof if the linear transforms and all entries in the data are constrained to be rational. ■

Remark 87. Due to [Lemma 86](#), we will interchangeably use the notation \mathcal{T}_R^{D-MI} and the phrase “finite unions of MILP-representable sets” (similarly, $\widehat{\mathcal{T}_R^{D-MI}}$ and “finite unions of sets in $\widehat{\mathcal{T}_R^{MI}}$ ”) without further comment in the remainder of this chapter.

Finally, we introduce concepts that are used in describing characterizations of these families of sets. The key concept used to articulate the “integrality” inherent in many of these families is the following.

Definition 88 (Monoid). *A set $C \subseteq \mathbb{R}^n$ is a monoid if for all $x, y \in C$, $x + y \in C$. A monoid is finitely generated if there exist $r^1, r^2, \dots, r^k \in \mathbb{R}^n$ such that*

$$C = \left\{ x : x = \sum_{i=1}^k \lambda_i r^i \text{ where } \lambda_i \in \mathbb{Z}_+; \forall i \in [k] \right\}.$$

We will often denote the right-hand side of the above as $\text{int cone}\{r^1, \dots, r^k\}$. Further, we say that C is a pointed monoid, if $\text{cone}(C)$ is a pointed cone. A finitely generated monoid is called rational if the generators r^1, \dots, r^k are all rational vectors.

Table 6.1: Families of sets under consideration.

| Notation | Family |
|----------------------------------|---|
| $\overline{\mathcal{T}}$ | bounded sets from the family \mathcal{T} |
| $\text{cl}(\mathcal{T})$ | the closures of sets in the family \mathcal{T} |
| $\mathcal{T}(\mathbb{Q})$ | the sets in family \mathcal{T} determined with rational data |
| \mathcal{T}_R | \mathcal{T} -representable sets |
| \mathcal{T}^{MI} | sets that are mixed-integer points in a polyhedron |
| $\widehat{\mathcal{T}^{MI}}$ | sets that are mixed-integer points in a generalized polyhedron |
| \mathcal{T}^{CBL} | continuous bilevel linear (CBL) sets |
| \mathcal{T}^{LC} | linear complementarity (LC) sets |
| \mathcal{T}^{BLP-UI} | bilevel linear polyhedral sets with integer upper level (BLP-UI) |
| \mathcal{T}^{D-MI} | sets that can be written as finite unions of sets in \mathcal{T}^{MI} |
| \mathcal{T}_R^{D-MI} | sets that can be written as finite unions of sets in \mathcal{T}_R^{MI} (see Lemma 86) |
| \mathcal{T}^{MIBL} | Mixed-integer bilevel sets |
| $\widehat{\mathcal{T}^{D-MI}}$ | sets that can be written as finite unions of sets in $\widehat{\mathcal{T}^{MI}}$ |
| $\widehat{\mathcal{T}_R^{D-MI}}$ | sets that can be written as finite unions of sets in $\widehat{\mathcal{T}_R^{MI}}$ (see Lemma 86) |
| \mathcal{T}^{IC} | integer cones |
| \mathcal{P} | finite unions of polyhedra |

In this chapter, we are interested in discrete monoids. A set S is discrete if there exists an $\varepsilon > 0$ such that for all $x \in S$, $B(x, \varepsilon) \cap S = \{x\}$. Not all discrete monoids are finitely generated. For example, the set $M = \{(0, 0)\} \cup \{x \in \mathbb{Z}^2 : x_1 \geq 1, x_2 \geq 1\}$ is a discrete monoid that is not finitely generated.

The seminal result from Jeroslow and Lowe [92], which we restate in [Theorem 108](#), shows that a rational MILP-representable set is the Minkowski sum of a finite union of rational polytopes and a rational finitely generated monoid. Finally, we define three families of functions that provide an alternative vocabulary for describing “integrality”; namely, Chvátal functions, Gomory functions and Jeroslow functions. These families derive significance here from their ability to articulate value functions of integer and mixed-integer programs (as seen in Blair and Jeroslow [30, 31], Blair [32], Blair and Jeroslow [33]).

Chvátal functions are defined recursively by using linear combinations and floor ($\lfloor \cdot \rfloor$) operators on other Chvátal functions, assuming that the set of affine linear functions are Chvátal functions. We formalize this using a binary tree construction as below. We adapt the definition from Basu et al. [21].²

Definition 89 (Chvátal functions Basu et al. [21]). *A Chvátal function $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ is constructed as follows. We are given a finite binary tree where each node of the tree is either: (i) a leaf node which corresponds to an affine linear function on \mathbb{R}^n with rational coefficients; (ii) has one child with a corresponding edge labeled by either $\lfloor \cdot \rfloor$ or a non-negative rational number; or (iii) has two children, each with edges labeled by a non-negative rational number. Start at the root node and recursively form functions corresponding to subtrees rooted at its children using the following rules.*

²The definition in Basu et al. [21] used $\lceil \cdot \rceil$ as opposed to $\lfloor \cdot \rfloor$. We make this change in this chapter to be consistent with Jeroslow and Blair’s notation. Also, what is referred to as “order” of the Chvátal function’s representation is called “ceiling count” in Basu et al. [21].

1. If the root has no children then it is a leaf node corresponding to an affine linear function with rational coefficients. Then ψ is the affine linear function.
2. If the root has a single child, recursively evaluating a function g , and the edge to the child is labeled as $\lfloor \cdot \rfloor$, then $\psi(x) = \lfloor g(x) \rfloor$. If the edge is labeled by a non-negative number α , define $\psi(x) = \alpha g(x)$.
3. Finally, if the root has two children, containing functions g_1, g_2 and edges connecting them labeled with non-negative rationals, a_1, a_2 , then $\psi(x) = a_1 g_1(x) + a_2 g_2(x)$.

We call the number of $\lfloor \cdot \rfloor$ operations in a binary tree used to represent a Chvátal function the order of this binary tree representation of the Chvátal function. Note that a given Chvátal function may have alternative binary tree representations with different orders.

Definition 90 (Gomory functions). A Gomory function G is the pointwise minimum of finitely many Chvátal functions. That is,

$$G(x) \quad := \quad \min_{1=1}^k \psi_i(x),$$

where ψ_i for $i \in [k]$ are all Chvátal functions.

Gomory functions are then used to build Jeroslow functions, as defined in Blair [32].

Definition 91 (Jeroslow function). Let G be a Gomory function. For any invertible matrix E , and any vector x , define $\lfloor x \rfloor_E := E \lfloor E^{-1} x \rfloor$. Let \mathcal{I} be a finite index set and let $\{E_i\}_{i \in \mathcal{I}}$ be a set of $n \times n$ invertible rational matrices indexed by \mathcal{I} , and $\{w_i\}_{i \in \mathcal{I}}$ be a set of rational vectors in \mathbb{R}^n indexed by \mathcal{I} . Then $J : \mathbb{R}^n \mapsto \mathbb{R}$ is a Jeroslow function if

$$J(x) \quad := \quad \max_{i \in \mathcal{I}} \left\{ G(\lfloor x \rfloor_{E_i}) + w_i^\top (x - \lfloor x \rfloor_{E_i}) \right\},$$

Remark 92. Note that we have explicitly allowed only rational entries in the data defining Chvátal, Gomory and Jeroslow functions. This is also standard in the literature since the natural setting for these functions and their connection to mixed-integer programming uses rational data.

Remark 93. Note that it follows from [Definitions 89 to 91](#), the family of Chvátal functions, Gomory functions and Jeroslow function are all closed under composition with affine functions and addition of affine functions.

A key result in Blair [\[32\]](#) is that the value function of a mixed-integer program with rational data is a Jeroslow function. This result allows us to express the lower-level optimality condition captured in the bilevel constraint [\(2.1c\)](#). This is a critical observation for our study of MIBL-representability.

We now have all the vocabulary needed to state our main results.

6.3 Main results

Our main results concern the relationship between the sets defined in [Table 6.1](#) and the novel machinery we develop to establish these relationships.

First, we explore bilevel sets with only continuous variables. We show that the sets represented by continuous bilevel constraints, linear complementarity constraints and polyhedral reverse convex constraints are all equivalent and equal to the family of finite unions of polyhedra.

Theorem 94. *The following holds:*

$$\mathcal{T}_R^{CBL} = \mathcal{T}_R^{LC} = \mathcal{T}_R^{PRC} = \mathcal{T}^{PRC} = \mathcal{P}.$$

The next result shows the difference between the equivalent families of sets in [Theorem 94](#) and the family of MILP-representable sets. The characterization of MILP-representable sets by Jeroslow and Lowe [\[92\]](#) (restated below as [Theorem 108](#)) is central to the argument here. Using this

characterization, we demonstrate explicit examples of sets that illustrate the lack of containment in these families.

Corollary 95. *The following holds:*

$$\begin{aligned}\mathcal{T}_R^{CBL} \setminus \mathcal{T}_R^{MI} &\neq \emptyset \text{ and} \\ \mathcal{T}_R^{MI} \setminus \mathcal{T}_R^{CBL} &\neq \emptyset.\end{aligned}$$

The next result shows that the lack of containment of these families of sets arises because of unboundedness.

Corollary 96. *The following holds:*

$$\overline{\mathcal{T}_R^{CBL}} = \overline{\mathcal{T}_R^{LC}} = \overline{\mathcal{T}_R^{PRC}} = \overline{\mathcal{P}} = \overline{\mathcal{T}_R^{MI}}.$$

Remark 97. The rational versions of [Theorem 94](#), [Corollary 95](#), [Corollary 96](#) all hold, i.e., one can replace all the sets in the statements by their rational counterparts. For example, the following version of [Theorem 94](#) holds:

$$\mathcal{T}_R^{CBL}(\mathbb{Q}) = \mathcal{T}_R^{LC}(\mathbb{Q}) = \mathcal{T}_R^{PRC}(\mathbb{Q}) = \mathcal{T}^{PRC}(\mathbb{Q}) = \mathcal{P}(\mathbb{Q}).$$

We will not explicitly prove the rational versions; the proofs below can be adapted to the rational case without any difficulty.

Our next set of results concern the representability by bilevel programs with integer variables. We show that, with integrality constraints in the upper level only, bilevel representable sets correspond to finite unions MILP-representable sets. Further allowing integer variables in lower level may yield sets that are not necessarily closed. However, we show that the closures of sets are again finite unions of MILP-representable sets. In contrast to [Remark 97](#), the rationality of the

data is an important assumption in this setting. This is to be expected: this assumption is crucial whenever one deals with mixed-integer points, as mentioned in the introduction.

Theorem 98. *The following holds:*

$$\mathcal{T}_R^{MIBL}(\mathbb{Q}) \supsetneq \text{cl}(\mathcal{T}_R^{MIBL}(\mathbb{Q})) = \mathcal{T}_R^{BLP-UI}(\mathbb{Q}) = \mathcal{T}_R^{D-MI}(\mathbb{Q}).$$

In fact, in the case of BLP-UI sets, the rationality assumption can be dropped; i.e.,

$$\mathcal{T}_R^{BLP-UI} = \mathcal{T}_R^{D-MI}.$$

Behind the proof of this result are two novel technical results that we believe have interest in their own right. The first concerns an algebra of sets that captures, to some extent, the inherent structure that arises when bilevel constraints and integrality interact. Recall that an algebra of sets is a collection of sets that is closed under taking complements and unions. It is trivial to observe that finite unions of generalized polyhedra form an algebra, i.e., a family that is closed under finite unions, finite intersections and complements. We show that a similar result holds even for finite unions of generalized mixed-integer representable sets.

Theorem 99. *The family of sets $\left\{ S \subseteq \mathbb{R}^n : S \in \widehat{\mathcal{T}_R^{D-MI}}(\mathbb{Q}) \right\}$ is an algebra over \mathbb{R}^n for any n .*

The connection of the above algebra to optimization is made explicit in the following theorem, which is used in the proof of [Theorem 98](#).

Theorem 100. *Let $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ be a Chvátal, Gomory, or Jeroslow function. Then (i) $\{x : \psi(x) \leq 0\}$ (ii) $\{x : \psi(x) \geq 0\}$ (iii) $\{x : \psi(x) = 0\}$ (iv) $\{x : \psi(x) < 0\}$ and (v) $\{x : \psi(x) > 0\}$ are elements of $\widehat{\mathcal{T}_R^{D-MI}}(\mathbb{Q})$.*

As observed below [Definition 91](#), the family of Jeroslow functions capture the properties of the bilevel constraint (2.1c) and thus [Theorem 100](#) proves critical in establishing [Theorem 98](#).

Finally, we also discuss the computational complexity of solving bilevel programs and its connections with representability. A key result in this direction is the following.

Theorem 101. *If S is a rational BLP-UI set, then the sentence “Is S non-empty?” is in \mathcal{NP} .*

Remark 102. In light of the [Theorems 98](#) and [101](#), we observe the following. While adding integrality constraints in the upper level improves the modeling power, it does not worsen the theoretical difficulty to solve such problems. We compare this with the results of [\[105\]](#) which says that if there are integral variables in the lower level as well, the problem is much harder (Σ_p^2 -complete). However, by [Theorem 98](#), this does not improve modeling power up to closures. This shows some of the subtle interaction between notions of complexity and representability.

6.4 Representability of continuous bilevel sets

The goal of this section is to prove [Theorem 94](#). We establish the result across several lemmata. First, we show that sets representable by polyhedral reverse convex constraints are unions of polyhedra and vice versa. Then we show three inclusions, namely $\mathcal{T}_R^{PRC} \subseteq \mathcal{T}_R^{CBL}$, $\mathcal{T}_R^{CBL} \subseteq \mathcal{T}_R^{LC}$ and $\mathcal{T}_R^{LC} \subseteq \mathcal{T}_R^{PRC}$. The three inclusions finally imply [Theorem 94](#).

Now we prove a useful technical lemma.

Lemma 103. *If \mathcal{T}^1 and \mathcal{T}^2 are two families of sets such that $\mathcal{T}^1 \subseteq \mathcal{T}_R^2$ then $\mathcal{T}_R^1 \subseteq \mathcal{T}_R^2$. Moreover, the rational version holds, i.e., $\mathcal{T}^1(\mathbb{Q}) \subseteq \mathcal{T}_R^2(\mathbb{Q})$ implies $\mathcal{T}_R^1(\mathbb{Q}) \subseteq \mathcal{T}_R^2(\mathbb{Q})$.*

Proof. Let $T \in \mathcal{T}_R^1$. This means there is a linear transform L^1 and $T^1 \in \mathcal{T}^1$ such that $T = L^1(T^1)$. Also, this means $T^1 \in \mathcal{T}_R^2$, by assumption. So there exists a linear transform L^2 and $T^2 \in \mathcal{T}^2$ such that $T^1 = L^2(T^2)$. So $T = L^1(T^1) = L^1(L^2(T^2)) = (L^1 \circ L^2)(T^2)$, proving the result. The rational version follows by restricting all linear transforms and sets to be rational. ■

We now establish the first building block of [Theorem 94](#).

Lemma 104. *The following holds:*

$$\mathcal{P} = \mathcal{T}^{PRC} = \mathcal{T}_R^{PRC}.$$

Proof. We start by proving the first equivalence. Consider the \supseteq direction first. Let $S \in \mathcal{T}^{PRC}$. Then $S = \{x \in \mathbb{R}^n : Ax \leq b, f_i(x) \geq 0 \text{ for } i \in [n']\}$ for some polyhedral convex functions f_i with k_i pieces each. First, we show that S is a finite union of polyhedra. Choose one halfspace from the definition of each of the functions $f_i(x) = \max_{j=1}^{k_i} \{\langle \alpha^{ij}, x \rangle - \beta_j\}$ (i.e., $\{x : \langle \alpha^{ij}, x \rangle - \beta_{ij} \geq 0\}$ for some j and each i) and consider their intersection. This gives a polyhedron. There are exactly $K = \prod_{i=1}^{n'} k_i$ such polyhedra. We claim that S is precisely the union of these K polyhedra, intersected with $\{x : Ax \leq b\}$ (clearly, the latter set is in \mathcal{P}). Suppose $x \in S$. Then $Ax \leq b$. Also since $f_i(x) \geq 0$, we have $\max_{j=1}^{k_i} \{\langle \alpha^{ij}, x \rangle - \beta_{ij}\} \geq 0$. This means for each i , there exists a j_i such that $\langle \alpha^{ij_i}, x \rangle - \beta_{ij_i} \geq 0$. The intersection of all such halfspaces is one of the K polyhedra defined earlier. Conversely, suppose x is in one of these K polyhedra (the one defined by $\langle \alpha^{ij_i}, x \rangle - \beta_{ij_i} \geq 0$ for $i \in [n']$) intersected with $\{x : Ax \leq b\}$. Then, $f_i(x) = \max_{j=1}^{k_i} \{\langle \alpha^{ij}, x \rangle - \beta_{ij}\} \geq \langle \alpha^{ij_i}, x \rangle - \beta_{ij_i} \geq 0$ and thus $x \in S$. This shows that $\mathcal{T}^{PRC} \subseteq \mathcal{P}$.

Conversely, suppose $P \in \mathcal{P}$ and is given by $P = \bigcup_{i=1}^k P_i$ and $P_i = \{x : A^i x \leq b^i\}$ where $b^i \in \mathbb{R}^{m_i}$. Let a_j^i refer to the j -th row of A^i and b_j^i the j -th coordinate of b^i . Let Ω be the Cartesian product of the index sets of constraints, i.e., $\Omega = \{1, \dots, m_1\} \times \{1, \dots, m_2\} \times \dots \times \{1, \dots, m_k\}$. For any $\omega \in \Omega$, define Now consider the following function:

$$f_\omega(x) = \max_{i=1}^k \{-\langle a_{\omega_i}^i, x \rangle + b_{\omega_i}^i\},$$

where ω_i denotes the index chosen in ω from the set of constraints in $A^i x \leq b^i$, $i = 1, \dots, k$. This construction is illustrated in [Figure 6.1](#). Let $T = \{x : f_\omega(x) \geq 0, \forall \omega \in \Omega\} \in \mathcal{T}^{PRC}$. We now claim

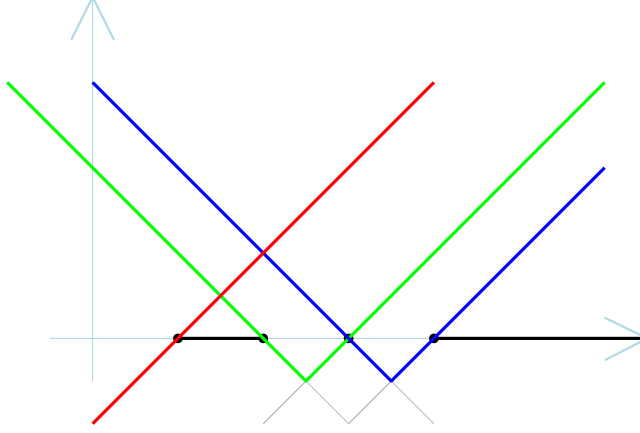


Figure 6.1: Representation of union of polyhedra $[1, 2] \cup \{3\} \cup [4, \infty)$ as a PRC set. The three colored lines correspond to three different the polyhedral convex functions. The points where each of those functions is non-negative precisely correspond to the region shown in black, the set which we wanted to represent in the first place.

that $P = T$. If $\bar{x} \in P$ then there exists an i such that $\bar{x} \in P_i$, which in turn implies that $b^i - A^i \bar{x} \geq 0$. However each of the f_ω contains at least one of the rows from $b^i - A^i \bar{x}$, and that is non-negative. This means each f_ω , which are at least as large as any of these rows, are non-negative. This implies $P \subseteq T$. Now suppose $\bar{x} \notin P$. This means in each of the k polyhedra P_i , at least one constraint is violated. Now consider the f_ω created by using each of these violated constraints. Clearly for this choice, $f_\omega(x) < 0$. Thus $\bar{x} \notin T$. This shows $T \subseteq P$ and hence $P = T$. This finally shows $\mathcal{P} \subseteq \mathcal{T}^{PRC}$. Combined with the argument in the first part of the proof, we have $\mathcal{T}^{PRC} = \mathcal{P}$.

Now consider the set \mathcal{P}_R . A linear transform of a union of finitely many polyhedra is a union of finitely many polyhedra. Thus $\mathcal{P}_R = \mathcal{P}$. But $\mathcal{P} = \mathcal{T}^{PRC}$ and so $\mathcal{T}_R^{PRC} = \mathcal{P}_R = \mathcal{P} = \mathcal{T}^{PRC}$, proving the remaining equivalence in the statement of the lemma. ■

Next, we show that any set representable using polyhedral reverse convex constraints is representable using continuous bilevel constraints. To achieve this, we give an explicit construction

of the bilevel set.

Lemma 105. *The following holds:*

$$\mathcal{T}_R^{PRC} \subseteq \mathcal{T}_R^{CBL}.$$

Proof. Suppose $S \in \mathcal{T}^{PRC}$. Then,

$$S = \{x \in \mathbb{R}^n : Ax \leq b, f_i(x) \geq 0 \text{ for } i \in [n']\}$$

for some n' , A , b and polyhedral convex functions $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ for $i \in [n']$. Further, let us explicitly write $f_i(x) = \max_{j=1}^{k_i} \{\langle \alpha^{ij}, x \rangle - \beta_{ij}\}$ for $j \in [k_i]$ for $i \in [n']$. Note for any i , $f_i(x) \geq 0$ if and only if $\{\langle \alpha^{ij}, x \rangle - \beta_{ij}\} \geq 0$ for each $j \in [k_i]$. Now, consider the following CBL set $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{n'}$ where $(x, y) \in S'$ if

$$\begin{aligned} Ax &\leq b \\ y &\geq 0 \\ y &\in \arg \max_y \left\{ -\sum_{i=1}^{n'} y_i : y_i \geq \langle \alpha^{ij}, x \rangle - \beta_{ij} \text{ for } j \in [k_i], i \in [n'] \right\}. \end{aligned} \quad (6.2)$$

A key observation here is that in (6.2), the condition $y_i \geq \langle \alpha^{ij}, x \rangle - \beta_{ij}$ for all $j \in [k_i]$ is equivalent to saying $y_i \geq f_i(x)$. Thus (6.2) can be written as $y \in \arg \min_y \left\{ \sum_{i=1}^{n'} y_i : y_i \geq f_i(x), i \in [n'] \right\}$. Since we are minimizing the sum of coordinates of y , this is equivalent to saying $y_i = f_i(x)$ for $i \in [n']$. However, we have constrained y to be non-negative. So, if $S'' = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{n'} \text{ such that } (x, y) \in S'\}$, it naturally follows that $S = S''$. Thus $S \in \mathcal{T}_R^{CBL}$, proving the inclusion $\mathcal{T}^{PRC} \subseteq \mathcal{T}_R^{CBL}$. The result then follows from [Lemma 103](#). \blacksquare

The next result shows that a given CBL-representable set can be represented as an LC-representable set. Again, we give an explicit construction.

Lemma 106. *The following holds:*

$$\mathcal{T}_R^{CBL} \subseteq \mathcal{T}_R^{LC}.$$

Proof. Suppose $S \in \mathcal{T}^{CBL}$. Let us assume that the parameters A, B, b, f, C, D, g that define S according to [Definition 12](#) has been identified. We first show that $S \in \mathcal{T}_R^{LC}$. Retaining the notation in [Definition 12](#), let $(x, y) \in S$. Then from [\(2.1c\)](#), y solves a linear program that is parameterized by x . The strong duality conditions (or the KKT conditions) can be written for this linear program are written below as [\(6.3b\)](#) - [\(6.3e\)](#) and hence S can be defined as (x, y) satisfying the following constraints for some λ of appropriate dimension:

$$Ax + By \leq b \tag{6.3a}$$

$$D^\top \lambda - f = 0 \tag{6.3b}$$

$$g - Cx - Dy \geq 0 \tag{6.3c}$$

$$\lambda \geq 0 \tag{6.3d}$$

$$(g - Cx - Dy)^\top \lambda = 0. \tag{6.3e}$$

Consider S' as the set of (x, y, λ) satisfying

$$b - Ax - By \geq 0$$

$$D^\top \lambda - f \geq 0$$

$$0 \leq \lambda \perp g - Cx - Dy \geq 0.$$

Clearly, $S' \in \mathcal{T}_R^{LC}$. Let $S'' = \{(x, y) : (x, y, \lambda) \in S'\}$. Clearly $S'' \in \mathcal{T}_R^{LC}$. We now argue that $S = S''$. This follows from the fact that Lagrange multipliers λ exist for the linear program in [\(2.1c\)](#) so that $(x, y, \lambda) \in S'$. Hence $\mathcal{T}^{CBL} \subseteq \mathcal{T}_R^{LC}$ follows and by [Lemma 103](#), the result follows. \blacksquare

Finally, to complete the cycle of containment, we show that a set representable using linear complementarity constraint can be represented as a polyhedral reverse convex set.

Lemma 107. *The following holds:*

$$\mathcal{T}_R^{LC} \subseteq \mathcal{T}_R^{PRC}.$$

Proof. Let $p, q \in \mathbb{R}$. Notice that $0 \leq p \perp q \geq 0 \iff p \geq 0, q \geq 0, \max\{-p, -q\} \geq 0$. Consider any $S \in \mathcal{T}^{LC}$. Then

$$S = \{x : 0 \leq x \perp Mx + q \geq 0, Ax \leq b\} = \{x : Ax \leq b, x \geq 0, Mx + q \geq 0, f_i(x) \geq 0\},$$

where $f_i(x) = \max\{-x_i, -[Mx + q]_i\}$. Clearly, each f_i is a polyhedral convex function and hence by definition S is a polyhedral reverse-convex set. This implies $S \in \mathcal{T}^{PRC}$. So, $\mathcal{T}^{LC} \subseteq \mathcal{T}^{PRC} \implies \mathcal{T}_R^{LC} \subseteq \mathcal{T}_R^{PRC}$. ■

With the previous lemmata in hand, we can now establish [Theorem 94](#).

Proof of [Theorem 94](#). Follows from [Lemmata 104](#) to [107](#). ■

We turn our focus to establishing [Corollary 95](#). This uses the following seminal result of Jeroslow and Lowe [\[92\]](#), which gives a geometric characterization of MILP-representable sets.

Theorem 108 (Jeroslow and Lowe [\[92\]](#)). *Let $T \in \mathcal{T}_R^{MI}(\mathbb{Q})$. Then,*

$$T = P + C$$

for some $P \in \overline{\mathcal{P}}(\mathbb{Q})$, $C \in \mathcal{T}^{IC}(\mathbb{Q})$.

We now give two concrete examples to establish [Corollary 95](#). First, a set which is CBL-representable but not MILP-representable and, second, an example of a set which is MILP-representable but not CBL-representable.

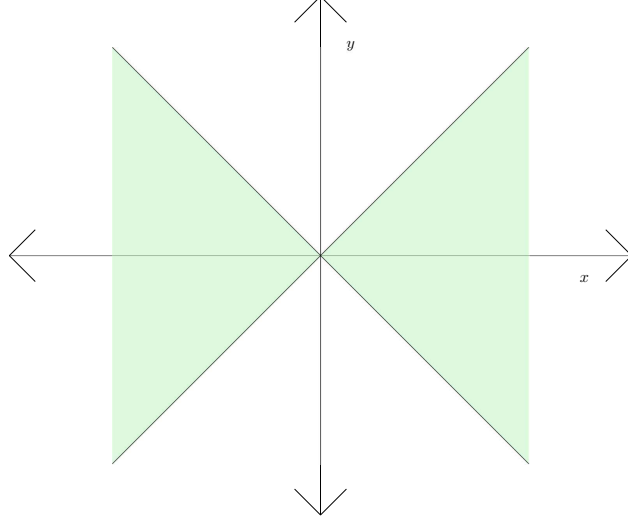


Figure 6.2: The set T used in the proof of [Corollary 95](#). Note that $T \in \mathcal{T}_R^{CBL} \setminus T_R^{MI}$

Proof of [Corollary 95](#). We construct a set $T \in \mathcal{T}_R^{CBL}$ as follows. Consider the following set $T' \in \mathcal{T}^{CBL}$ given by $(x, y, z_1, z_2) \in \mathbb{R}^4$ satisfying:

$$(y, z_1, z_2) \in \arg \min \left\{ z_1 - z_2 : \begin{array}{l} z_1 \geq x \\ z_1 \geq -x \\ z_2 \leq x \\ z_2 \leq -x \\ y \leq z_1 \\ y \geq z_2 \end{array} \right\}$$

with no upper-level constraints. Consider $T = \{(x, y) \in \mathbb{R}^2 : (x, y, z_1, z_2) \in T'\}$ illustrated in [Figure 6.2](#). Note that $T \in \mathcal{T}_R^{CBL}$. We claim $T \notin \mathcal{T}_R^{MI}$. Suppose it is. Then by [Theorem 108](#), T is the Minkowski sum of a finite union of polytopes and a monoid. Note that $\{(x, x) : x \in \mathbb{R}\} \subset T$ which implies $(1, 1)$ is an extreme ray and $\lambda(1, 1)$ should be in the integer cone of T for some $\lambda > 0$. Similarly $\{(-x, x) : x \in \mathbb{R}\} \subset T$ which implies $(-1, 1)$ is an extreme ray and $\lambda'(-1, 1)$ should be in the integer cone of T for some $\lambda' > 0$. Both the facts imply, for some $\lambda'' > 0$, the point $(0, \lambda'') \in T$.

But no such point is in T showing that $T \notin \mathcal{T}_R^{MI}$.

Conversely, consider the set of integers $\mathbb{Z} \subseteq \mathbb{R}$. Clearly $\mathbb{Z} \in \mathcal{T}_R^{MI}$ since it is the Minkowski sum of the singleton polytope $\{0\}$ and the integer cone generated by -1 and 1 . Suppose, by way of contradiction, that \mathbb{Z} can be expressed as a finite union of polyhedra (and thus in \mathcal{T}^{CBL} by [Theorem 94](#)). Then there must exist a polyhedron that contains infinitely many integer points. Such a polyhedron must be unbounded and hence have a non-empty recession cone. However, any such polyhedron has non-integer points. This contradicts the assumption that \mathbb{Z} is a finite union of polyhedra. ■

These examples show that the issue of comparing MILP-representable sets and CBL-representable sets arises in how these two types of sets can become unbounded. MILP-representable sets are unbounded in “integer” directions from a single integer cone, while CBL-representable sets are unbounded in “continuous” directions from potentially a number of distinct recession cones of polyhedra. Restricting to bounded sets removes this difference.

Proof of [Corollary 96](#). The first three equalities follow trivially from [Theorem 94](#). To prove that $\overline{\mathcal{P}} = \overline{\mathcal{T}_R^{MI}}$, observe from [Theorem 108](#) that any set in \mathcal{T}_R^{MI} is the Minkowski sum of a finite union of polytopes and a monoid. Observing that $T \in \mathcal{T}_R^{MI}$ is bounded if and only if the monoid is a singleton set containing only the zero vector, the equality follows. ■

6.5 Representability of mixed-integer bilevel sets

The goal of this section is to prove [Theorem 98](#). Again, we establish the result over a series of lemmata. In [Section 6.5.1](#), we show that the family of sets that are representable by bilevel linear polyhedral sets with integer upper-level variables is equal to the family of finite unions of MILP-

representable sets (that is, $\mathcal{T}_R^{BLP-UI} = \mathcal{T}_R^{D-MI}$). In [Section 6.5.2](#) we establish an important intermediate result to proving the rest of [Theorem 98](#), that the family $\widehat{\mathcal{T}_R^{D-MI}}$ is an algebra of sets; that is, it is closed under unions, intersections, and complements ([Theorem 99](#)). This pays off in [Section 6.5.4](#), where we establish the remaining containments of [Theorem 98](#) using the properties of this algebra and the characterization of value functions of mixed-integer programs in terms of Jeroslow functions, due to Blair [\[32\]](#).

6.5.1 Mixed-integer bilevel sets with continuous lower level

First, we show that any BLP-UI set is the finite union of MILP-representable sets.

Lemma 109. *The following holds:*

$$\mathcal{T}_R^{BLP-UI} \subseteq \mathcal{T}_R^{D-MI}$$

Moreover, the same inclusion holds in the rational case; i.e. $\mathcal{T}_R^{BLP-UI}(\mathbb{Q}) \subseteq \mathcal{T}_R^{D-MI}(\mathbb{Q})$.

Proof. Suppose $T \in \mathcal{T}_R^{BLP-UI}$. Then $T = \{x : x \in T'\} \cap \{x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L\}$, for some $T' \in \mathcal{T}^{CBL}$ and for some \mathcal{I}_L (we are abusing notation here slightly because we use the symbol x now to denote both the leader and follower variables in the bilevel program). By [Theorem 94](#), $T' \in \mathcal{P}$. Thus we can write $T' = \bigcup_{i=1}^k T'_i$ where each T'_i is a polyhedron. Now $T = \left(\bigcup_{i=1}^k T'_i\right) \cap \{x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L\} = \bigcup_{i=1}^k (\{x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L\} \cap T'_i)$ which is in \mathcal{T}_R^{D-MI} by definition. The result then follows from [Lemma 103](#). The rational version involving $\mathcal{T}_R^{BLP-UI}(\mathbb{Q})$ and $\mathcal{T}_R^{D-MI}(\mathbb{Q})$ holds by identical arguments restricting to rational data. ■

We are now ready to prove the reverse containment to [Lemma 109](#) and thus establish one of the equivalences in [Theorem 98](#).

Lemma 110. *The following holds:*

$$\mathcal{T}_R^{D-MI} \subseteq \mathcal{T}_R^{BLP-UI}.$$

Moreover, the same inclusion holds in the rational case; i.e. $\mathcal{T}_R^{BLP-UI}(\mathbb{Q}) \subseteq \mathcal{T}_R^{D-MI}(\mathbb{Q})$.

Proof. Let $T \in \mathcal{T}_R^{D-MI}$; then, by definition, there exist polyhedra \tilde{T}_i such that T is the linear image under the linear transform L of the mixed-integer points in the union of the \tilde{T}_i . Let $\tilde{T} = \bigcup_{i=1}^k \tilde{T}_i$ and so $T = \left\{ L(y, z) : (y, z) \in \tilde{T} \cap \left(\mathbb{R}^{n'} \times \mathbb{Z}^{n''} \right) \right\}$. By [Theorem 94](#), $\tilde{T} \in \mathcal{T}_R^{CBL}$. If the $z \in \mathbb{Z}^{n''}$ are all upper-level variables in the feasible region of the CBL that projects to \tilde{T} , then T is clearly in \mathcal{T}_R^{BLP-UI} . If, on the other hand, some z_i is a lower-level variable then adding a new integer upper-level variable w_i and upper-level constraint $w_i = z_i$ gives a BLP-UI formulation of a lifting of \tilde{T} , and so again $T \in \mathcal{T}_R^{BLP-UI}$. This proves the inclusion. The rational version involving $\mathcal{T}_R^{BLP-UI}(\mathbb{Q})$ and $\mathcal{T}_R^{D-MI}(\mathbb{Q})$ holds by identical arguments restricting to rational data. \blacksquare

6.5.2 The algebra $\widehat{\mathcal{T}_R^{D-MI}}$

We develop some additional theory for generalized polyhedra, as defined in [Section 6.2](#). The main result in this section is that the family of sets that are finite unions of sets representable by mixed-integer points in generalized polyhedra forms an algebra (in the sense of set theory). Along the way, we state some standard results from lattice theory and prove some key lemmata leading to the result. This is used in our subsequent proof in the representability of mixed-integer bilevel programs.

We first give a description of an arbitrary generalized polyhedron in terms of relatively open polyhedra. This allows us to extend properties we prove for relatively open polyhedra to generalized polyhedra.

Lemma 111. *Every generalized polyhedron is a finite union of relatively open polyhedra. If the generalized polyhedron is rational, then the relatively open polyhedra in the union can also be taken to be rational.*

Proof. We proceed by induction on the affine dimension d of the polyhedron. The result is true by definition for $d = 0$, where the only generalized polyhedron is a singleton, which is also a relatively open polyhedron. For higher dimensions, let $P = \{x : Ax < b, Cx \leq d\}$ be a generalized polyhedron. Without loss of generality, assume P is full-dimensional, because otherwise we can work in the affine hull of P . We then write

$$P = \left\{ x : \begin{array}{l} Ax < b \\ Cx < d \end{array} \right\} \cup \bigcup_i (P \cap \{x : \langle c^i, x \rangle = d_i\}).$$

The first set is a relatively open polyhedron and each of the sets in the second union is a generalized polyhedron of lower affine dimension, each of which are finite unions of relatively open polyhedra by the inductive hypothesis.

The rational version also goes along the same lines, by restricting the data to be rational. ■

From [123, Theorem 6.6], we obtain the following:

Lemma 112 (Rockafellar [123]). *Let $Q \subseteq \mathbb{R}^n$ be a relatively open polyhedron and L is any linear transformation. Then $L(Q)$ is relatively open. If Q and L are both rational, then $L(Q)$ is also rational.*

We now prove for relatively open polyhedra an equivalent result to the Minkowski-Weyl theorem for regular polyhedra.

Lemma 113. *Let $Q \subseteq \mathbb{R}^n$ be a relatively open polyhedron. Then $Q = P + R$ where P is a relatively open polytope and R is the recession cone $\text{rec cl}(Q)$. If Q is rational then P and R can also be taken to be rational.*

Proof. We first assume that $\dim(Q) = n$; otherwise, we can work in the affine hull of Q . Thus, we may assume that Q can be expressed as $Q = \{x \in \mathbb{R}^n : Ax < b\}$ for some matrix A and right hand side b .

We first observe that if L is the lineality space of $\text{cl}(Q)$, then $Q = \text{Proj}_{L^\perp}(Q) + L$ where $\text{Proj}_{L^\perp}(\cdot)$ denotes the projection onto the orthogonal subspace L^\perp to L . Moreover, $\text{Proj}_{L^\perp}(Q)$ is a relatively open polyhedron by [Lemma 112](#) and its closure is pointed (since we projected out the lineality space). Therefore, it suffices to establish the result for full-dimensional relatively open polyhedra whose closure is a pointed (regular) polyhedron. Henceforth, we will assume $\text{cl}(Q)$ is a pointed polyhedron.

Define $Q_\epsilon := \{x \in \mathbb{R}^n : Ax \leq b - \epsilon \mathbf{1}\}$ and observe that $Q_0 = \text{cl}(Q)$ and $Q = \cup_{0 < \epsilon \leq 1} Q_\epsilon$. Notice also that $\text{rec}(Q_\epsilon) = \text{rec}(\text{cl}(Q)) = R$ for all $\epsilon \in \mathbb{R}$ (they are all given by $\{r \in \mathbb{R}^n : Ar \leq 0\}$). Moreover, since $\text{cl}(Q)$ is pointed, Q_ϵ is pointed for all $\epsilon \in \mathbb{R}$. Also, there exists a large enough natural integer M such that the box $[-M, M]^n$ contains, in its interior, all the vertices of Q_ϵ for every $0 \leq \epsilon \leq 1$.

Define $P' := Q_0 \cap [-M, M]^n$ which is a regular polytope. By standard real analysis arguments, $\text{int}(P') = \text{int}(Q_0) \cap \text{int}([-M, M]^n)$. Thus, we have $\text{int}(P') + R \subseteq \text{int}(Q_0) + R = Q$. Next, observe that for any $\epsilon > 0$, $Q_\epsilon \subseteq \text{int}(P') + R$ because every vertex of Q_ϵ is in the interior of Q_0 , as well as in the interior of $[-M, M]^n$ by construction of M , and so every vertex of Q_ϵ is contained in $\text{int}(P')$. Since $Q = \cup_{0 < \epsilon \leq 1} Q_\epsilon$, we obtain that $Q \subseteq \text{int}(P') + R$. Putting both inclusions together, we obtain that $Q = \text{int}(P') + R$. Since $P := \text{int}(P')$ is a relatively open polytope, we have the desired result.

The rational version of the proof is along the same lines, where we restrict to rational data. ■

We also observe below that generalized polyhedra are closed under Minkowski sums (up to finite unions).

Lemma 114. *Let P, Q be generalized polyhedra, then $P + Q$ is a finite union of relatively open polyhedra. If P and Q are rational, then $P + Q$ is a finite union of rational relatively open polyhedra.*

Proof. By Lemma 111, it suffices to prove the result for relatively open polyhedra P and Q . The result then follows from the fact that $\text{relint}(P + Q) = \text{relint}(P) + \text{relint}(Q) = P + Q$, where the second equality follows from [123, Corollary 6.6.2].

The rational version follows along similar lines by using the rational version of Lemma 111. ■

We now prove a preliminary result on the path to generalizing Theorem 108 to generalized polyhedra.

Lemma 115. *Let $Q \subseteq \mathbb{R}^n \times \mathbb{R}^d$ be a rational generalized polyhedron. Then $Q \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ is a union of finitely many sets, each of which is the Minkowski sum of a relatively open rational polytope and a rational monoid.*

Proof. By Lemma 111, it is sufficient to prove the theorem where Q is relatively open. By Lemma 113, we can write $Q = P + R$ where P is a rational relatively open polytope and R is a cone generated by finitely many rational vectors. Set $T = P + X$, where $X = \left\{ \sum_{i=1}^k \lambda_i r^i : 0 \leq \lambda_i \leq 1 \right\}$ where r^i are the extreme rays of R whose coordinates can be chosen as integers. For $u + v \in Q = P + R$, where $u \in P$, $v \in R$; let $v = \sum_{i=1}^k \mu_i r^i$. Define $\gamma_i = \lfloor \mu_i \rfloor$ and $\lambda_i = \mu_i - \gamma_i$. So, $u + v = (u + \sum_{i=1}^k \lambda_i r^i) + \sum_{i=1}^k \gamma_i r^i$, where the term in parentheses is contained in T and since $\gamma_i \in \mathbb{Z}_+$, the second term is in a monoid generated by the extreme rays of R . Thus, we have $Q \cap (\mathbb{Z}^n \times \mathbb{R}^d) = (T \cap (\mathbb{Z}^n \times \mathbb{R}^d)) + \text{int cone}(r^1, \dots, r^k)$. Since T is a finite union of rational relatively open polytopes by Lemma 114, $T \cap (\mathbb{Z}^n \times \mathbb{R}^d)$ is a finite union of rational relatively open polytopes. ■

The following is an analog of Jeroslow and Lowe's fundamental result (Theorem 108) to the

generalized polyhedral setting.

Theorem 116. *The following are equivalent:*

- (1) $S \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$,
- (2) S is a finite union of sets, each of which is the Minkowski sum of a rational relatively open polytope and a rational finitely generated monoid, and
- (3) S is a finite union of sets, each of which is the Minkowski sum of a rational generalized polytope and a rational finitely generated monoid.

Proof. (1) \implies (2): Observe from [Lemma 112](#) that a (rational) linear transform of a (rational) relatively open polyhedron is a (rational) relative open polyhedron, and by definition of a (rational) monoid, a (rational) linear transform of a (rational) monoid is a (rational) monoid. Now from [Lemmata 86](#) and [115](#), the result follows.

(2) \implies (3): This is trivial since every (rational) relatively open polyhedron is a (rational) generalized polyhedron.

(3) \implies (1): This follows from the observation that the Minkowski sum of a rational generalized polytope and a rational monoid is a rational generalized mixed-integer representable set. A formal proof could be constructed following the proof of [Theorem 108](#) given in the original Jeroslow and Lowe [\[92\]](#) or [\[43, Theorem 4.47\]](#). These results are stated for the case of regular polyhedra but it is straightforward to observe that their proofs equally apply to the generalized polyhedra setting with only superficial adjustments. We omit those minor details for brevity. ■

Remark 117. Notice that the equivalence of (1) and (3) in [Theorem 116](#) is an analog of Jeroslow and Lowe's [Theorem 108](#). Moreover, the rationality assumption cannot be removed from [Lemma 115](#),

and hence cannot be removed from [Theorem 116](#). This is one of the places where the rationality assumption plays a crucial role; see also [Remark 124](#).

Now we prove that if we intersect sets within the family of generalized MILP-representable sets, then we remain in that family.

Lemma 118. *Let S and T be sets in \mathbb{R}^n where $S, T \in \widehat{\mathcal{T}}_R^{MI}$. Then $S \cap T \in \widehat{\mathcal{T}}_R^{MI}$. The rational version holds, i.e., one can replace $\widehat{\mathcal{T}}_R^{MI}$ by $\widehat{\mathcal{T}}_R^{MI}(\mathbb{Q})$ in the statement.*

Proof. Using the same construction in [Lemma 86](#), one can assume that there is a common ambient space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and a common linear transformation L such that

$$S = \left\{ L(x) : x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} : A^1 x \leq b^1, \hat{A}^1 x < \hat{b}^1 \right\}$$

$$T = \left\{ L(x) : x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} : A^2 x \leq b^2, \hat{A}^2 x < \hat{b}^2 \right\}.$$

Now, it is easy to see $S \cap T \in \widehat{\mathcal{T}}_R^{MI}$.

The rational version follows by restricting the linear transforms and data to be rational. ■

An immediate corollary of the above lemma is that the class $\widehat{\mathcal{T}}_R^{D-MI}$ is closed under finite intersections.

Lemma 119. *Let S and T be sets in \mathbb{R}^n where $S, T \in \widehat{\mathcal{T}}_R^{MI}$. Then $S \cap T \in \widehat{\mathcal{T}}_R^{D-MI}$. The rational version holds, i.e., one can replace $\widehat{\mathcal{T}}_R^{D-MI}$ by $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ in the statement.*

Proof. By [Lemma 86](#), $S = \bigcup_{i=1}^k S_i$ and $T = \bigcup_{i=1}^\ell T_i$ where $S_i, T_i \in \widehat{\mathcal{T}}_R^{MI}$. Now $S \cap T = \left(\bigcup_{i=1}^k S_i \right) \cap \left(\bigcup_{i=1}^\ell T_i \right) = \bigcup_{i=1}^k \bigcup_{j=1}^\ell (S_i \cap T_j)$. But from [Lemma 118](#), $S_i \cap T_j \in \widehat{\mathcal{T}}_R^{MI}$. Then the result follows from [Lemma 86](#).

The rational version follows from the rational versions of [Lemmata 86](#) and [118](#). ■

To understand the interaction between generalized polyhedra and monoids, we review a few standard terms and results from lattice theory. We refer the reader to [18, 43, 126] for more comprehensive treatments of this subject.

Definition 120 (Lattice). *Given a set of linearly independent vectors $d^1, \dots, d^r \in \mathbb{R}^n$, the lattice generated by the vectors is the set*

$$\Lambda = \left\{ x : x = \sum_{i=1}^r \lambda_i d^i, \lambda_i \in \mathbb{Z} \right\} \quad (6.4)$$

We call the vectors d^1, \dots, d^r as the generators of the lattice Λ and denote it by $\Lambda = Z(d^1, \dots, d^r)$.

Note that the same lattice Λ can be generated by different generators.

Definition 121 (Fundamental parallelepiped). *Given $\Lambda = Z(d^1, \dots, d^r) \subseteq \mathbb{R}^n$, we define the fundamental parallelepiped of Λ (with respect to the generators d^1, \dots, d^r) as the set*

$$\Pi_{\{d^1, \dots, d^r\}} := \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^r \lambda_i d^i, 0 \leq \lambda_i < 1 \right\}.$$

We prove the following two technical lemmata, which are crucial in proving that $\widehat{\mathcal{T}_R^{D-MI}}(\mathbb{Q})$ is an algebra. The lemmata prove that $(P + M)^c$, where P is a polytope and M is a finitely generated monoid, is in $\widehat{\mathcal{T}_R^{D-MI}}$ (and a corresponding rational version is true). The first lemma proves this under the assumption that M is generated by linearly independent vectors. The second lemma uses this preliminary results to prove it for a general monoid.

The proof of the lemma below where M is generated by linearly independent vectors is based on the following key observations.

- (i) If the polytope P is contained in the fundamental parallelepiped Π of the lattice generated by the monoid M , then the complement of $P + M$ is just $(\Pi \setminus P) + M$ along with everything outside $\text{cone}(M)$.

- (ii) The entire lattice can be written as a disjoint union of finitely many cosets with respect to an appropriately chosen sublattice. The sublattice is chosen such that its fundamental parallelepiped contains P (after a translation). Then combining the finitely many cosets with the observation in (i), we obtain the result.

The first point involving containment of P inside Π is needed to avoid any overlap between distinct translates of P in $P + M$. The linear independence of the generators of the monoid is important to be able to use the fundamental parallelepiped in this way. The proof also has to deal with the technicality that the monoid (and the lattice generated by it) need not be full-dimensional.

Lemma 122. *Let $M \subseteq \mathbb{R}^n$ be a monoid generated by a linearly independent set of vectors $\mathcal{M} = \{m^1, \dots, m^k\}$ and let P be a generalized polytope. Then $(P + M)^c \subseteq \mathcal{T}_R^{\widehat{D-MI}}$. Moreover, if P and M are both rational, then $(P + M)^c \subseteq \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.*

Proof. Suppose $k \leq n$. We now choose vectors $\tilde{m}^{k+1}, \dots, \tilde{m}^n$, a scaling factor $\alpha \in \mathbb{Z}_+$ and a translation vector $f \in \mathbb{R}^n$ such that the following all hold:

- (i) $\widetilde{\mathcal{M}} := \{m^1, \dots, m^k, \tilde{m}^{k+1}, \dots, \tilde{m}^n\}$ forms a basis of \mathbb{R}^n .
- (ii) $\{\tilde{m}^i\}_{i=k+1}^n$ are orthogonal to each other and each is orthogonal to the space spanned by \mathcal{M} .
- (iii) $f + P$ is contained in the fundamental parallelepiped defined by the vectors $\overline{\mathcal{M}} := \alpha\mathcal{M} \cup \{\tilde{m}^{k+1}, \dots, \tilde{m}^n\}$.

Such a choice is always possible because of the boundedness of P and by utilizing the Gram-Schmidt orthogonalization process. Since we are interested in proving inclusion in $\mathcal{T}_R^{\widehat{D-MI}}$, which is closed under translations, we can assume $f = 0$ without loss of generality.

Define $\widetilde{\Lambda} := Z(\widetilde{\mathcal{M}})$ and $\widetilde{M} := \text{int cone}(\widetilde{\mathcal{M}})$. Define $\overline{\Lambda} := Z(\overline{\mathcal{M}}) \subseteq \widetilde{\Lambda}$ and $\overline{M} := \text{int cone}(\overline{\mathcal{M}}) \subseteq$

\widetilde{M} . Moreover, linear independence of $\widetilde{\mathcal{M}}$ and $\overline{\mathcal{M}}$ implies that $\widetilde{M} = \widetilde{\Lambda} \cap \text{cone}(\widetilde{\mathcal{M}})$ and $\overline{M} = \overline{\Lambda} \cap \text{cone}(\overline{\mathcal{M}})$. All of these together imply

Claim 122.1: $\overline{M} = \overline{\Lambda} \cap \widetilde{M}$.

Proof of Claim 122.1: $\overline{M} = \overline{\Lambda} \cap \text{cone}(\overline{\mathcal{M}}) = \overline{\Lambda} \cap \text{cone}(\widetilde{\mathcal{M}}) = (\overline{\Lambda} \cap \widetilde{\Lambda}) \cap \text{cone}(\widetilde{\mathcal{M}}) = \overline{\Lambda} \cap (\widetilde{\Lambda} \cap \text{cone}(\widetilde{\mathcal{M}})) = \overline{\Lambda} \cap \widetilde{M}$, where the second equality follows from the fact that $\text{cone}(\overline{\mathcal{M}}) = \text{cone}(\widetilde{\mathcal{M}})$.

Claim 122.2: $\Pi_{\overline{\mathcal{M}}} + \overline{M} = \text{cone}(\widetilde{\mathcal{M}})$. Moreover, given any element in $x \in \text{cone}(\widetilde{\mathcal{M}})$, there exist unique $u \in \Pi_{\overline{\mathcal{M}}}$ and $v \in \overline{M}$ such that $x = u + v$.

Proof of Claim 122.2: Suppose $u \in \Pi_{\overline{\mathcal{M}}}$ and $v \in \overline{M}$, then both u and v are non-negative combinations of elements in $\overline{\mathcal{M}}$. So clearly $u + v$ is also a non-negative combination of those elements. This proves the forward inclusion. To prove the reverse inclusion, let $x \in \text{cone}(\widetilde{\mathcal{M}})$.

Then $x = \sum_{i=1}^k \lambda_i m^i + \sum_{i=k+1}^n \lambda_i \widetilde{m}^i$ where $\lambda_i \in \mathbb{R}_+$. But now we can write

$$x = \left(\sum_{i=1}^k \left\lfloor \frac{\lambda_i}{\alpha} \right\rfloor \alpha m^i + \sum_{i=k+1}^n \lfloor \lambda_i \rfloor \widetilde{m}^i \right) + \left(\sum_{i=1}^k \left(\frac{\lambda_i}{\alpha} - \left\lfloor \frac{\lambda_i}{\alpha} \right\rfloor \right) \alpha m^i + \sum_{i=k+1}^n (\lambda_i - \lfloor \lambda_i \rfloor) \widetilde{m}^i \right),$$

where the term in the first parentheses is in \overline{M} and the term in the second parentheses is in $\Pi_{\overline{\mathcal{M}}}$.

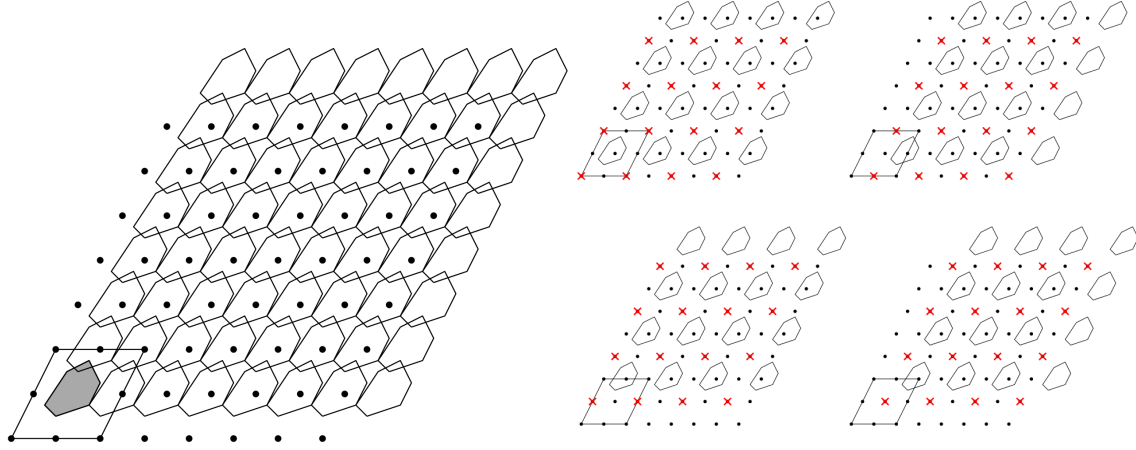
Uniqueness follows from linear independence arguments, thus proving the claim.

Claim 122.3: $(\Pi_{\overline{\mathcal{M}}} + \overline{M}) \setminus (P + \overline{M}) = (\Pi_{\overline{\mathcal{M}}} \setminus P) + \overline{M}$.

Proof of Claim 122.3: Note that, by construction, $P \subseteq \Pi_{\overline{\mathcal{M}}}$. By the uniqueness result in Claim 122.2, $(\Pi_{\overline{\mathcal{M}}} + u) \cap (\Pi_{\overline{\mathcal{M}}} + v) = \emptyset$ for $u, v \in \overline{M}$ and $u \neq v$. Thus we have, $x = u + v = u' + v'$ such that $u \in \Pi_{\overline{\mathcal{M}}}$, $u' \in P$, $v, v' \in \overline{M}$ implies $v = v'$. Then the claim follows.

Also, there exists a *finite* set S such that $\widetilde{\Lambda} = S + \overline{\Lambda}$ (for instance, S can be chosen to be $\Pi_{\overline{\mathcal{M}}} \cap \widetilde{\Lambda}$) [18, Theorem VII.2.5]. So we have

$$P + \widetilde{M} = P + (\widetilde{\Lambda} \cap \widetilde{M})$$



(a) P is shown in gray and the translates of P along \tilde{M} are shown. The fundamental parallelepiped $\Pi_{\bar{\Lambda}}$ is also shown to contain P .

(b) $P + ((s + \bar{\Lambda}) \cap \tilde{M})$ is shown for each $s \in S$. The red crosses correspond to the translation of \bar{M} along each $s \in S$. The union of everything in

Figure 6.3b is Figure 6.3a.

Figure 6.3: Intuition for the set S such that $\bar{\Lambda} + S = \tilde{\Lambda}$.

$$\begin{aligned}
&= \bigcup_{s \in S} \left(P + \left((s + \bar{\Lambda}) \cap \tilde{M} \right) \right) \\
&= \bigcup_{s \in S} \left(P + (\bar{\Lambda} \cap \tilde{M}) + s \right) \\
&= \bigcup_{s \in S} (P + \bar{M} + s), \tag{6.5}
\end{aligned}$$

where the last equality follows from Claim 122.1. The intuition behind the above equation is illustrated in Figure 6.3. We will first establish that $(P + \tilde{M})^c \in \mathcal{T}_R^{D-\widehat{MI}}$. By taking complements in (6.5), we obtain that $(P + \tilde{M})^c = \bigcap_{s \in S} (P + \bar{M} + s)^c$. But from Lemma 118, and from the finiteness of S , if we can show that $(P + \bar{M} + s)^c$ is in $\mathcal{T}_R^{D-\widehat{MI}}$ for every $s \in S$, then we would have established that $(P + \tilde{M})^c \in \mathcal{T}_R^{D-\widehat{MI}}$.

Since each of the finite $s \in S$ induce only translations, without loss of generality, we can only

consider the case where $s = 0$. Since we have $P + \overline{M} \subseteq \text{cone}(\widetilde{\mathcal{M}})$, we have

$$\begin{aligned} (P + \overline{M})^c &= \text{cone}(\widetilde{\mathcal{M}})^c \cup \left(\text{cone}(\widetilde{\mathcal{M}}) \setminus (P + \overline{M}) \right) \\ &= \text{cone}(\widetilde{\mathcal{M}})^c \cup \left((\Pi_{\overline{\mathcal{M}}} + \overline{M}) \setminus (P + \overline{M}) \right), \end{aligned} \quad (6.6)$$

which follows from Claim 122.2. Continuing from (6.6):

$$(P + \overline{M})^c = \text{cone}(\widetilde{\mathcal{M}})^c \cup \left((\Pi_{\overline{\mathcal{M}}} \setminus P) + \overline{M} \right),$$

which follows from Claim 122.3.

The first set $\text{cone}(\widetilde{\mathcal{M}})^c$ in (6.5.2) belongs to $\mathcal{T}_R^{\widehat{D-MI}}$ since the complement of a cone is a finite union of generalized polyhedra. In the second set $(\Pi_{\overline{\mathcal{M}}} \setminus P) + \overline{M}$, $\Pi_{\overline{\mathcal{M}}}$ and P are generalized polytopes, and hence $\Pi_{\overline{\mathcal{M}}} \setminus P$ is a finite union of generalized polytopes, $(\Pi_{\overline{\mathcal{M}}} \setminus P) + \overline{M}$ is a set in $\mathcal{T}_R^{\widehat{D-MI}}$. Thus, $(P + \overline{M})^c \in \mathcal{T}_R^{\widehat{D-MI}}$ (note that Lemma 86 shows that $\mathcal{T}_R^{\widehat{D-MI}}$ is closed under unions).

We now finally argue that $(P + M)^c$ belongs to $\mathcal{T}_R^{\widehat{D-MI}}$. Let $A^1 := (P + \widetilde{M})^c$.

For each vector \widetilde{m}^i for $i = k+1, \dots, n$ added to form $\widetilde{\mathcal{M}}$ from \mathcal{M} , define H^i as follows:

$$H^i = \{x : \langle \widetilde{m}^i, x \rangle \geq \|\widetilde{m}^i\|_2^2\}.$$

Now let $A^2 := \bigcup_{i=k+1}^n H^i$. Note that A^2 is a finite union of halfspaces and hence $A^2 \in \mathcal{T}_R^{\widehat{D-MI}}$.

We claim $(P + M)^c = A^1 \cup A^2$. This suffices to complete the argument since we have shown A^1 and A^2 are in $\mathcal{T}_R^{\widehat{D-MI}}$ and thus so is their union.

First, we show that $A^1 \cup A^2 \subseteq (P + M)^c$, i.e., $P + M \subseteq A_1^c \cap A_2^c$. Let $x \in P + M$. Since $M \subseteq \widetilde{M}$ we have $x \in P + \widetilde{M}$. Thus, $x \notin A^1$. Further, since $x \in P + M$ we may write $x = u + v$ with $u \in P$ and $v \in M$ where $u = \sum_{i=1}^k \mu_i \alpha m^i + \sum_{i=k}^n \mu_i \widetilde{m}^i$, $v = \sum_{j=1}^k \lambda_j m^j$ with $0 \leq \mu < 1$ and $\lambda_j \in \mathbb{Z}_+$, since $P \subseteq \Pi_{\overline{\mathcal{M}}}$. So for all i , $\langle \widetilde{m}^i, u \rangle = \mu_i \|\widetilde{m}^i\|_2^2 < \|\widetilde{m}^i\|_2^2$. This is because we have

\tilde{m}^i is orthogonal to every vector m^j and \tilde{m}^j for $i \neq j$. Hence, $\langle \tilde{m}^i, u + v \rangle < \|\tilde{m}^i\|_2^2$. This follows because \tilde{m}^i is orthogonal to the space spanned by the monoid $M \ni v$. Thus $x \notin A^2$. So we now have $P + M \subseteq A_1^c \cap A_2^c$ and so $A^1 \cup A^2 \subseteq (P + M)^c$.

Conversely, suppose $x \notin P + M$. If, in addition, $x \notin P + \widetilde{M}$ then $x \in A^1$ and we are done. However, if $x = u + v \in P + (\widetilde{M} \setminus M)$ with $u \in P$ and $v \in \widetilde{M} \setminus M$. This means $v = \sum_{j=1}^k \lambda_j m^j + \sum_{j=k+1}^n \lambda_j \tilde{m}^j$ with $\lambda_j \in \mathbb{Z}_+$ for all $j = 1, \dots, n$ and $\lambda_{\bar{j}} \geq 1$ for some $\bar{j} \in \{k+1, \dots, n\}$ and we can write $u = \sum_{j=1}^k \mu_j \alpha m^j + \sum_{j=k+1}^n \mu_j \tilde{m}^j$ with $0 \leq \mu \leq 1$. So $\langle \tilde{m}^i, u \rangle = \langle \tilde{m}^i, \mu^i \tilde{m}^i \rangle \geq 0$ and $\langle \tilde{m}^{\bar{j}}, v \rangle = \langle \tilde{m}^{\bar{j}}, \lambda_{\bar{j}} \tilde{m}^{\bar{j}} \rangle > \|\tilde{m}^{\bar{j}}\|_2^2$. So $u + v \in H^{\bar{j}} \subseteq A^2$. Thus we have the reverse containment and hence the result.

The rational version follows along similar lines. ■

Lemma 123. *Let $P \subseteq \mathbb{R}^n$ be a rational generalized polytope and $M \in \mathbb{R}^n$ be a rational, finitely generated monoid. Then $S = (P + M)^c \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.*

Proof. Define $C := \text{cone}(M)$. Consider a triangulation $C = \bigcup_i C_i$, where each C_i is simplicial. Now, $M_i := M \cap C_i$ is a monoid for each i (one simply checks the definition of a monoid) and moreover, it is a pointed monoid because C_i is pointed and $\text{cone}(M_i) = C_i$ since every extreme ray of C_i has an element of M_i on it. Observe that $M = \bigcup_i M_i$.

By Theorem 4, part 1) in Jeroslow [93], each of the M_i are finitely generated. By part 3) of the same theorem, each M_i can be written as $M_i = \bigcup_{j=1}^{w_i} (p^{i,j} + \overline{M}_i)$ for some finite vectors $p^{i,1}, \dots, p^{i,w_i} \subseteq M_i$, where \overline{M}_i is the monoid generated by the elements of M_i lying on the extreme rays of C_i . Now,

$$P + M = \bigcup_i (P + M_i) = \bigcup_i \bigcup_{j=1}^{w_i} (P + (p^{i,j} + \overline{M}_i)).$$

Thus by Lemma 119, it suffices to show that $(P + (p^{i,j} + \overline{M}_i))^c$ is in $\mathcal{T}_R^{\widehat{D-MI}}$. Since \overline{M}_i is generated

by linearly independent vectors, we have our result from [Lemma 122](#). ■

Remark 124. We do not see a way to remove the rationality assumption in [Lemma 123](#), because it uses Theorem 4 in Jeroslow [93] that assumes that the monoid is rational and finitely generated. This is the other place where rationality becomes a crucial assumption in the analysis (see also [Remark 117](#)).

Lemma 125. *If $S \in \widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ then $S^c \in \widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$.*

Proof. By [Theorem 116](#), S can be written as a finite union $S = \bigcup_{j=1}^{\ell} S_j$, with $S_j = P_j + M_j$, where P_j is a rational generalized polytope and M_j is a rational, finitely generated monoid. Observe $S_j^c = (P_j + M_j)^c$, which by [Lemma 123](#), is in $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$. Now by De Morgan's law, $S^c = \left(\bigcup_j S_j\right)^c = \bigcap_j S_j^c$. By [Lemma 119](#), $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ is closed under intersections, and we have the result. ■

Proof of Theorem 99. We recall that a family of sets \mathcal{F} is an algebra if the following two conditions hold. (i) $S \in \mathcal{F} \implies S^c \in \mathcal{F}$ and (ii) $S, T \in \mathcal{F} \implies S \cup T \in \mathcal{F}$. For the class of interest, the first condition is satisfied from [Lemma 125](#) and noting that the complement of finite unions is a finite intersection of the complements and that the family $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ is closed under finite intersections by [Lemma 119](#). The second condition is satisfied by [Lemma 86](#) which shows that $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ is the same as *finite unions* of sets in $\widehat{\mathcal{T}}_R^{MI}(\mathbb{Q})$. ■

6.5.3 Value function analysis

We now discuss the three classes of functions defined earlier, namely, Chvátal functions, Gomory functions, and Jeroslow functions, and show that their sublevel, superlevel and level sets are all elements of $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$. This is crucial for studying bilevel integer programs via a value function

approach to handling the lower-level problem using the following result.

Theorem 126 (Theorem 10 in Blair [32]). *For every rational mixed-integer program $\max_{x,y} \{c^\top x + d^\top y : Ax + By = b; (x, y) \geq 0, x \in \mathbb{Z}^m\}$ there exists a Jeroslow function J such that if the program is feasible for some b , then its optimal value is $J(b)$.*

We now show that the sublevel, superlevel, and level sets of Chvátal functions are all in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.

Lemma 127. *Let $\psi : \mathbb{R}^n \mapsto \mathbb{R}$ be a Chvátal function. Then (i) $\{x : \psi(x) \geq 0\}$, (ii) $\{x : \psi(x) \leq 0\}$, (iii) $\{x : \psi(x) = 0\}$, (iv) $\{x : \psi(x) < 0\}$, and (v) $\{x : \psi(x) > 0\}$ are all in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.*

Proof. (i) We prove by induction on the order $\ell \in \mathbb{Z}^+$ used in the binary tree representation of ψ (see Definition 89). For $\ell = 0$, a Chvátal function is a rational linear inequality and hence $\{x : \psi(x) \geq 0\}$ is a rational halfspace, which is clearly in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$. Assuming that the assertion is true for all orders $\ell \leq k$, we prove that it also holds for order $\ell = k + 1$. By [21, Theorem. 4.1], we can write $\psi(x) = \psi_1(x) + \lfloor \psi_2(x) \rfloor$ where ψ_1 and ψ_2 are Chvátal functions with representations of order no greater than k . Hence,

$$\begin{aligned} \{x : \psi(x) \geq 0\} &= \{x : \psi_1(x) + \lfloor \psi_2(x) \rfloor \geq 0\} \\ &= \{x : \exists y \in \mathbb{Z}, \psi_1(x) + y \geq 0, \psi_2(x) \geq y\}. \end{aligned}$$

We claim equivalence because, suppose \bar{x} is an element of the set in RHS with some $\bar{y} \in \mathbb{Z}$, \bar{y} is at most $\lfloor \psi_2(\bar{x}) \rfloor$. So if $\psi_1(\bar{x}) + \bar{y} \geq 0$, we immediately have $\psi_1(\bar{x}) + \lfloor \psi_2(\bar{x}) \rfloor \geq 0$ and hence \bar{x} is in the set on LHS. Conversely, if \bar{x} is in the set on LHS, then choosing $\bar{y} = \lfloor \bar{x} \rfloor$ satisfies all the conditions for the sets in RHS, giving the equivalence. Finally, observing that the RHS is an intersection of sets which are already in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by the induction hypothesis and the fact that $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is an algebra by Theorem 99, we have the result.

(ii) By similar arguments as (i), the statement is true for $\ell = 0$. For positive ℓ , we proceed by induction using the same construction. Now,

$$\begin{aligned}\{x : \psi(x) \leq 0\} &= \{x : \psi_1(x) + \lfloor \psi_2(x) \rfloor \leq 0\} \\ &= \{x : \exists y \in \mathbb{Z}, \psi_1(x) + y \leq 0, \psi_2(x) \geq y, \psi_2(x) < y + 1\}.\end{aligned}$$

The last two conditions along with integrality on y ensures $y = \lfloor \psi_2(x) \rfloor$. Note that $\{x : \psi_2(x) - y \geq 0\}$ is in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by (i). Similarly $\{x : \psi_2(x) - y - 1 \geq 0\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$. Since $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is an algebra (cf. [Theorem 99](#)), its complement is in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ and hence we have $\{x : \psi_2(x) < y + 1\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$. Finally from the induction hypothesis, we have $\{x : \psi_1(x) + y \leq 0\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$. Since $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is closed under finite intersections, the result follows.

(iii) Set defined by (iii) is an intersection of sets defined in (i) and (ii).

(iv)-(v) Sets defined here are complements of sets defined in (i)-(ii). ■

Lemma 128. *Let $G : \mathbb{R}^n \mapsto \mathbb{R}$ be a Gomory function. Then (i) $\{x : G(x) \geq 0\}$, (ii) $\{x : G(x) \leq 0\}$, (iii) $\{x : G(x) = 0\}$, (iv) $\{x : G(x) < 0\}$, (v) $\{x : G(x) > 0\}$ are all in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.*

Proof. Let $G(x) = \min_{i=1}^k \psi_i(x)$, where each ψ_i is a Chvátal function.

(i) Note that $\{x : G(x) \geq 0\} = \bigcap_{i=1}^k \{x : \psi_i(x) \geq 0\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ since each individual set in the finite intersection is in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by [Lemma 127](#) and $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is closed under intersections by [Lemma 119](#).

(ii) Note that $G(x) \leq 0$ if and only if there exists an i such that $\psi_i(x) \leq 0$. So $\{x : G(x) \leq 0\} = \bigcup_{i=1}^k \{x : \psi_i(x) \leq 0\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ since each individual set in the finite union is in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by [Lemma 127](#), and $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is an algebra by [Theorem 99](#).

(iii) This is the intersection of sets described in (i) and (ii).

(iv)-(v) Sets defined here are complements of sets defined in (i)-(ii). ■

Lemma 129. *Let $J : \mathbb{R}^n \mapsto \mathbb{R}$ be a Jeroslow function. Then (i) $\{x : J(x) \geq 0\}$, (ii) $\{x : J(x) \leq 0\}$, (iii) $\{x : J(x) = 0\}$, (iv) $\{x : J(x) < 0\}$, and (v) $\{x : J(x) > 0\}$ are all in $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$.*

Proof. (i) Let $J(x) = \max_{i \in \mathcal{I}} G(\lfloor x \rfloor_{E_i}) + w_i^\top (x - \lfloor x \rfloor_{E_i})$ be a Jeroslow function, where G is a Gomory function, \mathcal{I} is a finite set, $\{E_i\}_{i \in \mathcal{I}}$ is set of rational invertible matrices indexed by \mathcal{I} , and $\{w_i\}_{i \in \mathcal{I}}$ is a set of rational vectors indexed by \mathcal{I} . Since we have a maximum over finitely many sets, from the fact that $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is an algebra, it suffices to show $\{x : G(\lfloor x \rfloor_E) + w^\top (x - \lfloor x \rfloor_E) \geq 0\} \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ for arbitrary E , w and Gomory function G . Observe that

$$\left\{x : G(\lfloor x \rfloor_E) + w^\top (x - \lfloor x \rfloor_E) \geq 0\right\} = \text{Proj}_x \left\{ (x, y^1, y^2, y^3) : \begin{array}{l} G(y^1) + y^2 \geq 0 \\ y^1 = \lfloor x \rfloor_E \\ y^2 = \langle w, y^3 \rangle \\ y^3 = x - y^1 \end{array} \right\}$$

and the set being projected in the right hand side above is equal to the following intersection

$$\begin{aligned} & \{(x, y^1, y^2, y^3) : G(y^1) + y^2 \geq 0\} \\ \cap & \{(x, y^1, y^2, y^3) : E^{-1}y^1 = \lfloor E^{-1}x \rfloor\} \\ \cap & \{(x, y^1, y^2, y^3) : y^2 = \langle w, y^3 \rangle\} \\ \cap & \{(x, y^1, y^2, y^3) : y^3 = x - y^1\}. \end{aligned}$$

Since each of the sets in the above intersection belong to $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by [Lemmata 127](#) and [128](#), and $\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ is an algebra by [Theorem 99](#), we obtain the result.

(ii) As in (i), since we have maximum over finitely many sets, from the fact that $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ is an algebra ([Theorem 99](#)), it suffices to show $\{x : G(\lfloor x \rfloor_E) + w^\top(x - \lfloor x \rfloor_E) \leq 0\} \in \widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$ for arbitrary E , w and Gomory function G . The same arguments as before pass through, except for replacing the \geq in the first constraint with \leq .

(iii) This is the intersection of sets described in (i) and (ii).

(iv)-(v) Sets defined here are complements of sets defined in (i)-(ii). ■

Proof of [Theorem 100](#). Follows from [Lemmata 127](#) to [129](#). ■

6.5.4 General mixed-integer bilevel sets

We start by quoting an example from Köppe et al. [[98](#)] showing that the MIBL set need not even be a closed set. This is the first relation in [Theorem 98](#), showing a strict containment.

Lemma 130. [[98](#), *Example 1.1*] *The following holds:*

$$\mathcal{T}_R^{MIBL} \setminus \text{cl}(\mathcal{T}_R^{MIBL}) \neq \emptyset.$$

Proof. The following set T is in $\mathcal{T}_R^{MIBL} \setminus \text{cl}(\mathcal{T}_R^{MIBL})$:

$$T = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y \in \arg \min_y \{y : y \geq x, 0 \leq y \leq 1, y \in \mathbb{Z}\} \right\}.$$

By definition, $T \in \mathcal{T}_R^{MIBL}$. Observe that the bilevel constraint is satisfied only if $y = \lceil x \rceil$. So $T = \{(0, 0)\} \cup ((0, 1] \times \{1\})$. So T is not a closed set. Observing that every set in $\text{cl}(\mathcal{T}_R^{MIBL})$ is closed, $T \notin \text{cl}(\mathcal{T}_R^{MIBL})$. ■

Now we prove a lemma that states that rational MIBL-representable sets are in $\widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$.

Lemma 131. *The following holds: $\mathcal{T}_R^{MIBL}(\mathbb{Q}) \subseteq \widehat{\mathcal{T}}_R^{D-MI}(\mathbb{Q})$.*

Proof. Recall from [Definition 12](#) an element S of $\mathcal{T}^{MIBL}(\mathbb{Q})$ consists of the intersection $S^1 \cap S^2 \cap S^3$ (with rational data). From [Theorem 126](#), S^2 can be rewritten as $\{(x, y) : f^\top y \geq J(g - Cx)\}$ for some Jeroslow function J . Thus, from [Lemma 129](#), $S^2 \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$. Moreover, $S^1, S^3 \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ since they are either rational polyhedra or mixed-integer points in rational polyhedra. Thus, $S = S^1 \cap S^2 \cap S^3 \in \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$ by [Theorem 99](#), proving the inclusion. This shows that $\mathcal{T}^{MIBL}(\mathbb{Q}) \subseteq \mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})$, and by [Lemma 103](#) the result follows. ■

[Lemma 131](#) gets us close to showing $\text{cl}(\mathcal{T}_R^{MIBL}(\mathbb{Q})) \subseteq \mathcal{T}_R^{D-MI}(\mathbb{Q})$, as required in [Theorem 98](#). Indeed, we can immediately conclude from [Lemma 131](#) that $\text{cl}(\mathcal{T}_R^{MIBL}(\mathbb{Q})) \subseteq \text{cl}(\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q}))$. The next few results build towards showing that $\text{cl}(\mathcal{T}_R^{\widehat{D-MI}}) = \mathcal{T}_R^{D-MI}$, and consequently $\text{cl}(\mathcal{T}_R^{\widehat{D-MI}}(\mathbb{Q})) = \mathcal{T}_R^{D-MI}(\mathbb{Q})$. The latter is intuitive since closures of generalized polyhedra are regular polyhedra. As we shall see, the argument is a bit more delicate than this simple intuition. We first recall a couple of standard results on the closure operation $\text{cl}(\cdot)$.

Lemma 132. *If $S_1, \dots, S_k \in \mathbb{R}^n$ then $\text{cl}(\bigcup_{i=1}^n S_i) = \bigcup_{i=1}^n \text{cl}(S_i)$.*

Proof. Note $S_j \subseteq \bigcup_i S_i$ for any $j \in [k]$. So $\text{cl}(S_j) \subseteq \text{cl}(\bigcup_i S_i)$. So, by union on both sides over all $j \in [n]$, we have that the RHS is contained in LHS. Conversely, observe that the RHS is a closed set that contains every S_i . But by definition, LHS is the inclusion-wise *smallest* closed set that contains all S_i . So the LHS is contained in the RHS, proving the lemma. ■

Lemma 133. *Let A, B be sets such that A is a finite union of convex sets, $\text{cl}(A)$ is compact and B is closed. Then $\text{cl}(A + B) = \text{cl}(A) + B$.*

Proof. For the inclusion \supseteq , we refer to Corollary 6.6.2 in Rockafellar [\[123\]](#), which is true for arbitrary convex sets A, B . The result naturally extends using [Lemma 132](#) even if A is a finite union of convex sets. For the reverse inclusion, consider $z \in \text{cl}(A + B)$. This means, there exist infinite sequences

$\{x^i\}_{i=1}^\infty \subseteq A$ and $\{y^i\}_{i=1}^\infty \subseteq B$, such that the sequence $\{x^i + y^i\}_{i=1}^\infty$ converges to z . Now, since $\text{cl}(A) \supseteq A$, $x^i \in \text{cl}(A)$ and since $\text{cl}(A)$ is compact, there exists a subsequence, which has a limit in $\text{cl}(A)$. Without loss of generality, let us work only with such a subsequence x^i and the limit as x . Now from the fact that each $y^i \in B$, B is a closed set and the sequence y^i converges to $z - x$, we can say $z - x \in B$. This proves the result, as we wrote z as a sum of $x \in \text{cl}(A)$ and $z - x \in B$. ■

Lemma 134. *The following holds: $\text{cl}(\widehat{\mathcal{T}_R^{D-MI}}) = \mathcal{T}_R^{D-MI}$. Moreover, $\text{cl}(\widehat{\mathcal{T}_R^{D-MI}}(\mathbb{Q})) = \mathcal{T}_R^{D-MI}(\mathbb{Q})$.*

Proof. The \supseteq direction is trivial because sets in \mathcal{T}_R^{D-MI} are closed and a regular polyhedron is a type of generalized polyhedron. For the \subseteq direction, let $S \in \text{cl}(\widehat{\mathcal{T}_R^{D-MI}})$; that is, $S = \text{cl}(\bigcup_{i=1}^k (P_i + M_i))$ for some P_i that are finite unions of generalized polytopes and M_i that are finitely generated monoids. By Lemma 132, this equals $\bigcup_{i=1}^k \text{cl}(P_i + M_i)$. Observe that P_i is a finite union of generalized *polytopes* and are hence bounded. Thus their closures are compact. Also M_i are finitely generated monoids and are hence closed. Thus, by Lemma 133, we can write this is equal to $\bigcup_{i=1}^k \text{cl}(P_i) + M_i$. But by Theorem 108, each of these sets $\text{cl}(P_i) + M_i$ is in \mathcal{T}_R^{MI} . Thus, their finite union is in \mathcal{T}_R^{D-MI} by Lemma 86.

The rational version follows by assuming throughout the proof that the generalized polytopes and monoids are rational. ■

The following is then an immediate corollary of Lemmata 131 and 134.

Corollary 135. *The following holds: $\text{cl}(\mathcal{T}_R^{MIBL}(\mathbb{Q})) \subseteq \mathcal{T}_R^{D-MI}(\mathbb{Q})$.*

We are now ready to prove the main result of the section.

Proof of Theorem 98. The strict inclusion follows from Lemma 130. The equalities $\mathcal{T}_R^{BLP-UI}(\mathbb{Q}) = \mathcal{T}_R^{D-MI}(\mathbb{Q})$ and $\mathcal{T}_R^{BLP-UI} = \mathcal{T}_R^{D-MI}$ are obtained from Lemmata 109 and 110. For the

equality $\text{cl}(\mathcal{T}_R^{MIBL}(\mathbb{Q})) = \mathcal{T}_R^{D-MI}(\mathbb{Q})$, the inclusion \supseteq follows from the equality $\mathcal{T}_R^{BLP-UI}(\mathbb{Q}) = \mathcal{T}_R^{D-MI}(\mathbb{Q})$ and the facts that BLP-UI sets are MIBL sets and sets in \mathcal{T}_R^{D-MI} are closed. The reverse inclusion is immediate from [Corollary 135](#). \blacksquare

Finally, we prove [Theorem 101](#).

Proof of Theorem 101. Following the notation in [Definition 12](#), $S = S^1 \cap S^2 \cap S^3$. Since $|I_F| = 0$, $(x, y) \in S^2$ if and only if there exists a $\lambda \leq 0$ such that $D^\top \lambda = f$ and $\lambda^\top (g - Cx) = f^\top y$. However, this is same as checking if there exists (x, y) such that $(x, y) \in S^1 \cap S^3 \cap \{(x, y) : \exists \lambda \leq 0, D^\top \lambda = f, \lambda^\top g - f^\top y \leq \lambda^\top Cx\}$ is non-empty. But this set is a set of linear inequalities, integrality requirements along with *exactly one* quadratic inequality. From Del Pia et al. [\[51\]](#), this problem is in \mathcal{NP} . \blacksquare

By reduction from regular integer programming, we obtain this corollary.

Corollary 136. *Bilevel linear programs with rational data integrality constraints only in the upper level is \mathcal{NP} -complete.*

6.6 Conclusion

In this chapter, we give a characterization of the types of sets that are representable by feasible regions of mixed-integer bilevel linear programs. In the case of bilevel linear programs with only continuous variables, the characterization is in terms of the finite unions of polyhedra. In the case of mixed-integer variables, the characterization is in terms of finite unions of generalized mixed-integer linear representable sets. Interestingly, the family of finite unions of polyhedra and the family of finite unions of generalized mixed-integer linear representable sets are both algebras of sets. The parallel between these two algebras suggests that generalized mixed-integer linear representable sets

are the “right” geometric vocabulary for describing the interplay between projection, integrality, and bilevel structures. We are hopeful that the algebra properties of finite unions of generalized mixed-integer linear representable sets may prove useful in other contexts.

There remain important algorithmic and computational questions left unanswered in this study. For instance, is there any computational benefit of expressing a bilevel program in terms of disjunctions of (generalized) mixed-integer linear representable sets? Are there problems that are naturally expressed as disjunctions of generalized mixed-integer linear representable sets that can be solved using algorithms for mixed-integer bilevel programming (such as the recent work by Fischetti et al. [72], Wang and Xu [138], etc.)? In the case of continuous bilevel linear programs, its equivalence with unions of polyhedra suggests a connection to solving problems over unions of polyhedra, possibly using the methodology of Balas et al. [15]. The connection between bilevel constraints and linear complementarity also suggest a potential for interplay between the computational methods of both types of problems.

6.7 Acknowledgements

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Chapter 7

Conclusion and Future work

In this chapter, we briefly summarize the key results obtained throughout the dissertation. And then we mention some questions and directions of research that follow as a natural continuation of the dissertation.

7.1 Concluding remarks

In this dissertation, we discussed the modeling power obtained by a careful introduction of nonconvex constraints. In particular, we discussed three families of nonconvex problems, namely, mixed-integer programs, complementarity problems, and mixed-integer bilevel programs. We were interested in quantifying the trade-off between their modeling capability versus the additional computational burden in solving these problems.

Following the characterization of the modeling power of mixed-integer programs by Jeroslow and Lowe [92] as a finite union of polytopes plus a finitely generated monoid, we characterize the modeling power of linear complementarity problems as finite union of polyhedra, continuous bilevel problems also as finite union of polyhedra, and that of mixed-integer bilevel programs as finite

union of sets that are representable by mixed-integer programs ([Theorem 98](#)).

Besides theoretically characterizing their modeling power, we employed one of these nonconvexities — complementarity nonconvexity — to estimate Nash-equilibrium in two real-life problems. The first problem where this was applied was that of understanding the behavior of the natural gas market in North America (Refer [Section 4.2](#)). Since there are uncertainties involved in the parameters of such a problem, we were interested in the covariance of the resulting vector containing the Nash-equilibrium of the problem. Since solving the integral to obtain such a covariance is computationally expensive, we developed a method based on Taylor series expansion, that approximates the covariance of the solution of a stochastic complementarity problem. We then used these constraints to understand the effects of governmental policies on cereal export in Ethiopia (Refer [Section 4.3](#)).

Now with theoretical quantification of the modeling capacity of these problems and a more practical evidence in the applicability of these problems, we are interested in efficient ways to solve these problems. For problems with integer constraints, a significant part of the modern research is on generating valid inequalities for the corner polyhedron defined in [eq. \(3.9\)](#) of [Definition 27](#). Our contribution in this area is to give a parametrized family of valid inequalities for the corner polyhedron that are simultaneously strong and can be computed efficiently. For mixed-integer bilevel programs, we give an algorithm that can solve the problem in polynomial time provided the number of integer variables and the number of follower variables is fixed.

7.2 Future work

Machine learning is one of the fastest growing fields that has benefitted extensively from the developments in the field of optimization. In a world where data-driven decisions become more and

more important, and the quantity of data keeps growing, fast algorithms to make optimal decisions are crucial. While the first decade of the 21st century used methods from nonlinear optimization, structured nonconvex optimization is becoming more and more important in making machine learning algorithms more efficient. For example, Bennett et al. [24] use bilevel programming to do hyperparameter optimization for machine learning problems. Kunapuli [101] uses mathematical programs with complementarity constraints for support-vector regression and bilevel optimization for missing value imputation. Similarly, Bertsimas and Dunn [25] use mixed-integer programs to build a classification tree that has high accuracy on standardized test sets for machine learning algorithms. The methods, algorithms and structural results introduced in this dissertation for handling structured nonconvex optimization problems can provide useful directions in machine learning research, as described in the subsections below.

7.2.1 Theoretical and Structural questions

One of the reasons that a linear program is considered an “easy” problem to solve is because of the availability of a dual problem, which satisfies the so-called strong duality property and the fact that the dual problem is also a linear program of comparable size. There are some ideas of duality for mixed-integer programs too, ideas from which are used in speeding up the existing algorithms to solve mixed-integer programs.

Mixed-integer bilevel programs, on the other hand, do not have a well-developed duality theory. It would be both theoretically interesting and algorithmically useful to derive dual programs for bilevel programs. This could result in deriving cutting planes valid for the feasible region of the mixed-integer bilevel programs and eventually faster algorithms to solve them.

Another area with developing interest is to identify the role of sparsity in mixed-integer

programs. Very large instances of mixed-integer programs are routinely solved when the matrices defining the constraints are sparse. While tree-width based approaches [68, for example] partially explain this behavior, we still do not fully characterize the set of easy instances, for any structured sparsity forms. These are of greater interest, since many hard (NP -complete) problems can be posed as reasonably sparse MIPs and many problems arising from real-life applications are sparse. Quantifying the structure in such instances and methodically tailoring algorithms to solve these problems is another potential extension of the work in this dissertation.

7.2.2 Theoretical questions driven by applications

Given the extensive representability properties of mixed-integer bilevel programs described in [Section 6.3](#), characterizing the class of optimization problems that can be approximated *well* with mixed-integer bilevel programs becomes important. If structured optimization problems that result from machine learning applications can be posed as a mixed-integer bilevel program of a reasonable size that could be solved efficiently, this can lead to a new paradigm in solving optimization problems originating from machine learning problems approximately to global optimality. Bertsimas and Dunn [25] give evidence that solving a learning problem to global optimality increases the performance of the decision-tree algorithm for classification. This approach of solving machine learning based optimization problems approximately to global optimality will contrast the current approach of solving this class of optimization problems to first-order stationary points in some cases and local minima in some cases. While the literature on if such an approach would be useful is sparse, mixed-integer bilevel programs might serve as one of the first methods which can help solve a significant subset of machine learning-inspired optimization problems to global optimality and understand the usefulness of such an approach.

7.2.3 Novel and extended applications

The need to control climate change is driving policies geared towards using cleaner forms of energy across the globe. Such policies pose new challenges in modeling the energy markets. With the presence of carbon tax, energy obtained from cleaner means could be distinguished from energy obtained by burning fossil fuels. This is coupled with the fact that increasing number of domestic consumers resort to using solar panels for domestic consumption, creating significant differences in the demand structure for energy. These changes require updated models reflecting consumer behavior to evaluate policies efficiently. More interestingly, this could require us to model a game between various countries who act as multiple “leaders” and the energy market players in each of the countries who act as “followers”, a problem that can be posed as *an equilibrium problem with equilibrium constraints (EPEC)*. Developing algorithms to solve EPECs and hence analyze the markets would be a useful extension of the work in this dissertation.

The DECO2 model ([Section 4.3](#)) is capable of answering long-term policy analysis questions. For example, the impact of global warming on cropping patterns in Ethiopia can be assessed using DECO2. A useful extension of the model is to incorporate machine learning model predictions. Machine-learning models that can predict global demand for teff, coffee and other export goods with high accuracy can help in designing flexible government policies.

Extending the model to accommodate uncertainty in futuristic scenarios and modeling the actions of relatively short-sighted players can quantify the merits of long-term planning in economies. Such a model would be a better representation of the reality and hence will enable us to understand the response of the market to external stimuli much better.

Appendix A

Standard Notations and Symbols

| | |
|--------------------|---|
| \emptyset | The empty set |
| \mathbb{R} | Set of real numbers |
| \mathbb{Q} | Set of rational numbers |
| \mathbb{Z} | Set of integers |
| \mathbb{N} | Set of natural numbers $1, 2, \dots$ |
| $[n]$ | Natural numbers $1, \dots, n$. |
| $\text{conv}(S)$ | Convex hull of the set S |
| $\text{cone}(S)$ | Conic hull of the set S |
| $\text{aff}(S)$ | Affine hull of the set S |
| $\text{int}(S)$ | Interior of the set S |
| $\text{relint}(S)$ | Relative interior of the convex set S |
| $x \perp y$ | Asserting $x^T y = 0$ |

Appendix B

Equations for North American natural gas model

In this chapter of the appendix, we provide the set of optimization problems along with their KKT conditions which that are a part of the North American natural gas model.

In this formulation, we assume we have a set of suppliers P , consumers C and a pipeline operator. The players are located in a set of nodes N , and some of them are connected by pipelines A . Let also say that $P_n \subseteq P$, $C_n \subseteq C$ are located in node $n \in N$. Let A_n be the pipelines connected to node n . The symbols used here are explained in Table B.1 to B.3. Most of the analysis closely follow [69] and [60]. Random parameters are denoted by an (ω) beside them. The implementation of this problem is made available in https://github.com/ssriram1992/Stoch_Aprx_cov.

B.1 Producer's problem

$$\text{Maximize } \sum_Y df_y \left\{ \sum_C Q_{pcny}^C \pi_{cy} - \text{Gol} (Q_{pny}^P, \text{CAP}_{py}^P) \right\}$$

Table B.1: Sets

| Set | Explanation | Set | Explanation |
|-----|------------------|---------|--|
| P | Set of suppliers | A | Set of pipeline connections(arcs) |
| C | Set of consumers | A_n^+ | Set of arcs from node n on which natural gas flows out |
| N | Set of nodes | A_n^- | Set of arcs from node n on which natural gas flows in |
| Y | Set of periods | | |

Table B.2: Symbols - Variables

| | Symbol | Explanation |
|-------------------|--------------|---|
| Quantities | Q_{pcny}^C | Quantity produced by p in n to send to c in year y |
| | Q_{pny}^P | Total quantity produced by p in year y |
| | Q_{pay}^A | Total quantity p choses to send by arc a in year y |
| | Q_{ay}^A | Total quantity sent by a during year y |
| Prices | π_{cy} | Unit price paid by consumer C in year Y |
| | π_{ay}^A | Unit price of sending natural gas through a during year y |
| Capacity | X_{py}^P | Production expansion in year y for supplier p |
| | X_{ay}^A | Transportation capacity expansion in year y for arc a |
| | CAP_{py}^P | Production capacity for supplier p in year y |
| | CAP_{ay}^A | Transportation capacity for arc a in year y |

Table B.3: Symbols - Parameters

| | Symbol | Explanation |
|-------------------|---------------------|--|
| Quantities | \hat{Q}_{p0} | Initial capacity of production for supplier p |
| | \hat{Q}_{a0} | Initial capacity of transportation for pipeline a |
| Prices | $\pi_{py}^{XP}(xi)$ | Price of capacity expansion for supplier p |
| | $\pi_{ay}^{XA}(xi)$ | Price of capacity expansion for transportation arc a |
| Losses | $L_{py}^P(\omega)$ | Percentage loss in production by supplier p in year y |
| | $L_{ay}^A(\omega)$ | Percentage loss in transportation via arc a in year y |
| | α^P | Availability fraction of the production capacity |
| Consumer | $E_{cy}^C(\omega)$ | Intercept of the demand curve for consumer c in year y |
| | $D_{cy}^C(\omega)$ | Slope of the demand curve for consumer c in year y |
| | df_y | Discount Factor for year y |

$$\left. -\pi_{py}^{XP}(xi)X_{py}^P - \sum_{A_n^+} \pi_{ay}^A Q_{pay}^A \right\} \quad (B.1)$$

subject to

$$Q_{pcny}^C, Q_{pny}^P, Q_{pay}^A \geq 0$$

$$X_{py}^P, CAP_{py}^P \geq 0$$

$$Q_{pny}^P \leq \alpha^P CAP_{py}^P \quad (\delta_{py}^1) \quad (B.2a)$$

$$CAP_{py}^P = \hat{Q}_{p0} + \sum_{i=1}^y X_{pi}^P \quad (\delta_{py}^2) \quad (B.2b)$$

$$\begin{aligned} \sum_{C_n} Q_{pcny}^C + \sum_{A_n^+} Q_{pay}^A &= Q_{pny}^P(1 - L_{py}^P(\omega)) \\ &+ \sum_{A_n^-} Q_{pay}^A(1 - L_{ay}^A(\omega)) \quad (\delta_{pny}^3) \end{aligned} \quad (B.2c)$$

where

$$\begin{aligned} \text{Gol}(\cdot) &= (l_{py}^P(\omega) + g_{py}^P(\omega))Q_{pny}^P + q_{py}^P(\omega) (Q_{pny}^P)^2 \\ &+ g_{py}^P(\omega)(CAP_{py}^P - Q_{pny}^P) \log \left(1 - \frac{Q_{pny}^P}{CAP_{py}^P} \right) \end{aligned} \quad (B.3)$$

B.2 Pipeline operator's problem

$$\text{Maximize } \sum_Y df_y \left\{ \sum_A Q_{ay}^A (\pi_{ay}^A - \gamma_{ya}^A(\omega)) - \pi_{ay}^{XA}(xi)X_{ay}^A \right\} \quad (B.4)$$

subject to

$$Q_{ay}^A, X_{ay}^A, CAP_{ay}^A \geq 0$$

$$Q_{ay}^A \leq CAP_{ay}^A \quad (\delta_{ay}^5) \quad (B.5a)$$

$$CAP_{ay}^A = \hat{Q}_{a0} + \sum_{i=1}^y X_{ai}^A \quad (\delta_{ay}^6) \quad (B.5b)$$

B.3 Consumer

$$\pi_{cy} = E_{cy}^C(\omega) + D_{cy}^C(\omega) \sum_P Q_{pcny}^C \quad (\pi_{cy}) \quad (B.6)$$

It can be shown that the above said optimization problems are all convex with non-empty interior. Hence the Karush-Kuhn Tucker conditions (KKT conditions) are necessary and sufficient for optimality. The KKT conditions are presented below and they form the equations for the complementarity problem along with the constraints above.

B.4 KKT to Producer's problem

$$-df_y \pi_{cy} + \delta_{pny}^3 \geq 0 \quad (Q_{pcny}^C) \quad (B.7a)$$

$$df_y \pi_{py}^{XP}(xi) - \sum_{i=1}^y \delta_{pi}^2 \geq 0 \quad (X_{py}^P) \quad (B.7b)$$

$$df_y \pi_{ay}^A + \left(\mathbb{I}_{a \in A_n^+} - \mathbb{I}_{a \in A_n^-} (1 - L_{ay}^A(\omega)) \right) \delta_{pny}^3 \geq 0 \quad (Q_{pay}^A) \quad (B.7c)$$

$$df_y \frac{\partial \text{Gol}}{\partial Q_{pny}^P} + \delta_{py}^1 - \delta_{pny}^3 (1 - L_{py}^P(\omega)) \geq 0 \quad (Q_{pny}^P) \quad (B.7d)$$

$$df_y \frac{\partial \text{Gol}}{\partial \text{CAP}_{py}^P} + \alpha^P \delta_{py}^2 - \delta_{py}^1 \geq 0 \quad (\text{CAP}_{py}^P) \quad (B.7e)$$

B.5 KKT to Pipeline operator's problem

$$-df_y \pi_{ay}^A + \gamma_{ya}^A(\omega) + \delta_{ay}^5 \geq 0 \quad (Q_{ay}^A) \quad (B.8a)$$

$$df_y \pi_{ay}^{XA}(xi) - \sum_{i=1}^y \delta_{ai}^6 \geq 0 \quad (X_{ay}^A) \quad (B.8b)$$

$$\delta_{ay}^6 - \delta_{ay}^5 \geq 0 \quad (\text{CAP}_{ay}^A) \quad (B.8c)$$

B.6 Market clearing condition

$$Q_{ay}^A = \sum_P Q_{pay}^A \quad (\pi_{ay}^A) \quad (\text{B.9})$$

Appendix C

N-dimensional Sparse array implementation

A general purpose Python class has been implemented to handle a sparse *ndarray* object. The class is a generalization of the scipy class *coo_matrix* which stores the array coordinates of each non-zero element in the array. We now describe the details of the implementation. A continuously updated version of the class can be found at <https://github.com/ssriram1992/ndsparse>.

C.1 Initialization

The n-dimensional sparse array (`coo_array`) can be initialized by any of the following methods.

- A `tuple`, which initializes the sparse array of the shape mentioned in the `tuple` and with zeros everywhere.
- A dense `ndarray` which will be converted and stored as a `coo_array`.

- A `matrix` of positions and a 1 dimensional `array` of values where the matrix contains the positions of the non-zero elements and the vector containing the non-zero values of those positions. In this case, the shape of the `coo_array` would be the smallest `ndarray` that can store all the elements given. Optionally a `tuple` containing the shape of the `ndarray` can be given explicitly.
- Another `coo_array` whose copy is to be created.

C.2 Methods

The following methods and attributes are available in the `coo_array`.

- `print(coo_array)` will result in printing the location of each of the non-zero elements of the array and their values.
- `coo_array.flush(tol = 1e-5)` will result in freeing the space used in storing any zero-elements or elements lesser than the tolerance, `tol`. Such numbers typically arise out arithmetic operations on `coo_array` or poor initialization.
- `coo_array.size()` returns the number of non-zero elements in the `coo_array`.
- `coo_array.shape` returns the shape of the underlying dense matrix.
- `coo_array.add_entry(posn, val)` and `coo_array.set_entry(posn, val)` both add a new non-zero element with the given value at the given position. The difference however is that `set_entry()` checks if a non-zero value already exists at the mentioned position, and if yes, it overwrites it instead of creating a duplicate entry with the same coordinates. This search makes `set_entry()` slower compared to `add_entry()` which assumes that the previous value

or the position is zero. Thus `add_entry()` could potentially cause duplicates and ambiguity, if an illegal input is given. However in case the input is ensured to be legal, `add_entry()` is much faster.

- `coo_array.get_entry(posn)` returns the value at the given position.
- `coo_array.swapaxes(axis1,axis2)` is a higher dimensional generalization of matrix transposes where the dimensions that have to be swapped can be chosen.
- `coo_array.remove_duplicate_at(posn,func=0)` checks if there are multiple values defined for a single position in the sparse array. If yes, they are replaced by a single entry containing the scalar valued defined by `func` or passes them to a function defined in `func` and stores the returned value. Passing a function for the argument `func` is incredibly useful in performing arithmetic operations on `coo_array`.
- `coo_array.todense()` returns a dense version of the `coo_array`.
- `coo_array.iterate()` returns an iterable over the non-zero positions and values in the `coo_array`.

The above class is used extensively to handle high-dimensional sparse arrays in NANGAM to approximate the solution covariance according to [Algorithm 4](#).

Appendix D

Equations for DECO2

In this chapter, we present the optimization problems used in the DECO2 model for the Ethiopian food market.

D.1 Crop Producer

D.1.1 Problem

A crop producer maximizes her profit, which is the benefit given production and unit price minus the cost of cropping area, expanding the area, and converting cropping area for different crop types.

$$\begin{aligned} \text{Maximize} \quad &: \sum_{\substack{y \in Y \\ h \in H}} \text{df}_y \left\{ \sum_{f \in C} \left(\left(\sum_n \pi_{yhnf}^F \mathbf{Q}_{yhznf}^{FD} \right) - \mathcal{C}_{yhzf}^F \mathbf{A}_{yhzf}^F - \frac{1}{2} \mathcal{C}_{yhn}^{\text{change}} \left(\mathbf{A}_{yhnf}^F - \mathbf{A}_{(y-1)hzf}^F \right)^2 \right) \right. \\ &\quad \left. - \mathcal{C}_{yhn}^{\text{conv}} \sum_{f \in C} \left(\mathbf{A}_{yhnf}^F - \mathbf{A}_{(y-1)hzf}^F \right) \right\} \end{aligned} \quad (\text{D.1})$$

where Y is year; L is crop season; C is crop type; n is node/region; z is adaptation zone; F in superscript stands for food products; df_y is a discount factor for year y 's present value; π is unit price of production; \mathbf{Q} is production quantity; C is cost; A is cropping area.

The stakeholder is subject to the constraints for $f \in C$

$$\mathbf{Q}_{yhzf}^F, \mathbf{A}_{yhzf}^F \geq 0 \quad (\text{D.2a})$$

$$\mathbf{A}_z \geq \sum_{\substack{f \in C \\ h \in H}} \mathbf{A}_{yhzf}^F \quad (\delta_{yz}^1) \quad (\text{D.2b})$$

$$\mathbf{Q}_{yhzf}^F \leq \mathcal{Y}_{yhzf} \mathbf{A}_{yhzf}^F \quad (\delta_{yhzf}^2) \quad (\text{D.2c})$$

Linking yield with climate yield factor:

$$\mathcal{Y}_{yhnf} = \mathbf{a}_{yhzf} \text{CYF}_{yhzf} (\pi_{yhzf}^F)^e \quad (\mathcal{Y}_{yhnf}) \quad (\text{D.2d})$$

where \mathcal{Y}_{yhnf} is yield; \mathbf{a}_{yhzf} is a adjustment factor; CYF_{yhzf} is Climate Yield Factor; e is crop own elasticity.

$$\mathbf{Q}_{yhznf}^{FD} = \Psi_{nz} \mathbf{Q}_{yhzf}^F \quad (\delta_{yhznf}^{18}) \quad (\text{D.2e})$$

$$(\text{D.2f})$$

D.1.2 KKT Conditions

These KKT conditions for this problem hold for $f \in C$

$$\delta_{yhzf}^2 - \sum_n \Psi_{nz} \delta_{yhznf}^{18} \geq 0 \quad (\mathbf{Q}_{yhzf}^F) \quad (\text{D.3a})$$

$$\left. \begin{aligned} & \delta_{yz}^1 + \text{df}_y \left(\mathcal{C}_{yhzf}^F + \mathcal{C}_{yhz}^{\text{conv}} - \mathcal{C}_{(y+1)hz}^{\text{conv}} + \mathcal{C}_{yhz}^{\text{change}} \mathbf{A}_{yhnf}^F + \mathcal{C}_{(y+1)hz}^{\text{change}} \mathbf{A}_{yhnf}^F \right) \\ & - \delta_{yhzf}^2 \mathcal{Y}_{yhzf} - \text{df}_y \left(\mathcal{C}_{yhz}^{\text{change}} \mathbf{A}_{(y-1)hzf}^F + \mathcal{C}_{(y+1)hz}^{\text{change}} \mathbf{A}_{(y+1)hzf}^F \right) \end{aligned} \right\} \geq 0 \quad (\mathbf{A}_{yhzf}^F) \quad (\text{D.3b})$$

$$\delta_{yhznf}^{18} - \text{df}_y \pi_{yhnf}^F \geq 0 \quad (\mathbf{Q}_{yhznf}^{FD}) \quad (\text{D.3c})$$

where the sign of the last term depends upon whether f is the main crop or the secondary crop used for rotation.

D.2 Livestock producer

D.2.1 Problem

$$\text{Maximize} \quad : \quad \sum_{\substack{y \in Y \\ h \in H \\ f \notin C}} \text{df}_y \left(\left(\sum_n \pi_{yhnf}^F \mathbf{Q}_{yhnf}^{FD} \right) - \mathcal{B}_{yhz} \mathcal{C}_{yhz}^{\text{cow}} \right) \quad (\text{D.4})$$

where $\mathcal{C}_{yhz}^{\text{cow}}$ is the general management cost of cow.

The stakeholder is subject to the constraints for $f \notin C$

$$\mathcal{B}_{yhz}^{\text{buy}}, \mathcal{B}_{yhz}, \mathcal{B}_{yhz}^{\text{slg}} \geq 0 \quad (\text{D.5a})$$

$$\mathbf{Q}_{yhzf}^F, \mathbf{Q}_{yhz}^L \geq 0$$

Production quantity is yield of milk and beef per bovine times the number of bovine owned or slaughtered:

$$\mathbf{Q}_{yhzf}^F \leq \mathcal{Y}_{yhzf} \mathcal{B}_{yhz} \quad (f = \text{Milk}) \quad (\delta_{yhzf}^2)$$

$$\mathbf{Q}_{yhzf}^F \leq \mathcal{Y}_{yhzf} \mathcal{B}_{yhz}^{\text{slg}} \quad (f = \text{Beef}) \quad (\delta_{yhzf}^2)$$

Number of Bovine slaughtered should not exceed the number owned:

$$\mathcal{B}_{yhz}^{\text{slg}} \leq \mathcal{B}_{yhz} \quad (\delta_{yhz}^4) \quad (\text{D.5b})$$

Quantity balance of Bovine:

$$\mathcal{B}_{yhz} \leq (1 + k - \kappa) \mathcal{B}_{(y-1)hz} - \mathcal{B}_{(y-1)hz}^{\text{slg}} \quad (\pi_{yhz}^{\text{cow}}) \quad (\text{D.5c})$$

where k is the birth rate; κ is the death rate.

Number of Bovine should not be fewer than the natural deaths:

$$\mathcal{B}_{yhz}^{\text{slg}} \geq \kappa_{yhz}^{\text{death}} \mathcal{B}_{yhz} \quad (\delta_{yhz}^9) \quad (\text{D.5d})$$

Number of Bovine owned should be above certain level for reproduction (cannot slaughter all the livestock):

$$\begin{aligned} \mathcal{B}_{yhz} &\geq \mathcal{B}_z^{\text{herd}} & (\delta_{yhz}^{10}) & (\text{D.5e}) \\ \mathbf{Q}_{yhznf}^{FD} &= \Psi_{nz} \mathbf{Q}_{yhnf}^F & (\delta_{yhznf}^{18}) \end{aligned}$$

D.2.2 KKT Conditions

$$\left. \begin{aligned} \text{df}_y \mathcal{C}_{yhz}^{\text{cow}} - \delta_{yhzf}^2 \mathcal{Y}_{yhnf} - \delta_{yhz}^4 + \kappa_{yhz}^{\text{death}} \delta_{yhz}^9 \\ - \delta_{yhz}^{10} + \pi_{yhn}^{\text{cow}} - (1 + k - \kappa) \pi_{(y+1)hn}^{\text{cow}} \end{aligned} \right\} \geq 0 \quad (f = \text{Milk}) \quad (\mathcal{B}_{yhz}) \quad (\text{D.6a})$$

$$\delta_{yhzf}^2 - \text{df}_y \Psi_{nz} \delta_{yhznf}^{18} \geq 0 \quad (\mathbf{Q}_{yhzf}^F)$$

$$\delta_{yhz}^4 - \delta_{yhz}^9 - \delta_{yhzf}^2 \mathcal{Y}_{yhnf} + \pi_{(y+1)hz}^{\text{cow}} \geq 0 \quad (f = \text{Beef}) \quad (\mathcal{B}_{yhz}^{\text{slg}}) \quad (\text{D.6b})$$

$$\delta_{yhznf}^{18} - \text{df}_y \pi_{yhnf}^F \geq 0 \quad (\mathbf{Q}_{yhznf}^{FD}) \quad (\text{D.6c})$$

D.2.3 Adaptation zones to administrative regions

Here we convert production data from adaptation zone level to administrative region level to model transportation. These are indeed market clearing equations.

$$\mathbf{Q}_{yhnf}^{D_b} = \sum_z \mathbf{Q}_{yhznf}^{FD} \quad (\pi_{yhnf}^F) \quad (\text{D.7a})$$

D.3 Distribution

D.3.1 Problem

$$\text{Maximize} \quad : \quad \sum_{\substack{y \in Y \\ h \in H \\ f \in F}} \text{df}_y \left\{ \sum_{n \in N} \left(\mathbf{Q}_{yhnf}^{D_s} \pi_{yhnf}^W - \mathbf{Q}_{yhnf}^{D_b} \pi_{yhnf}^F \right) - \sum_{r \in \mathbb{R}} \mathcal{C}_{yhrf}^R \mathbf{Q}_{yhrf}^D \right\} \quad (\text{D.8})$$

where D in superscript stands for distributor; W in superscript stands for warehouse (storage manager); R stands for road and represents the linkages between nodes; \mathcal{C}_{yhrf}^R is the transportation cost per unit of quantity distributed.

The stakeholder is subject to the constraints

$$\mathbf{Q}_{yhnf}^{D_b}, \mathbf{Q}_{yhrf}^D, \mathbf{Q}_{yhnf}^{D_s} \geq 0 \quad (\text{D.9a})$$

The quantity purchased at one node n plus the quantity being distributed into n should be greater than total quantity sold and shipped out at n :

$$\mathbf{Q}_{yhnf}^{D_b} + \sum_{r \in R_{\text{in}}} \mathbf{Q}_{yhrf}^D \geq \mathbf{Q}_{yhnf}^{D_s} + \sum_{r \in R_{\text{out}}} \mathbf{Q}_{yhrf}^D \quad (\delta_{yhnf}^6) \quad (\text{D.9b})$$

Road capacity constraint per food item:

$$\mathbf{Q}_{yhrf}^D \leq \mathbf{Q}_{yhrf}^{R, \text{CAP}} \quad (\delta_{yhrf}^7) \quad (\text{D.9c})$$

Road capacity constraint:

$$\sum_f \mathbf{Q}_{yhrf}^D \leq \mathbf{Q}_{yhr}^{R, \text{CAP}} \quad (\delta_{yhr}^{16}) \quad (\text{D.9d})$$

Note that equation (3.2c) will be active depending upon the **FoodDistrCap** setting.

D.3.2 KKT Conditions

Representing s_r and d_r as the source and destination nodes of the transport system $r \in R$, we have the following KKT conditions.

$$\mathbf{df}_y \pi_{yhnf}^F - \delta_{yhnf}^6 \geq 0 \quad (\mathbf{Q}_{yhnf}^{D_b}) \quad (\text{D.10a})$$

$$\delta_{yhrf}^7 + \delta_{yhr}^{16} + \mathbf{df}_y \mathcal{C}_{yhrf}^R + \delta_{yhsrf}^6 - \delta_{yhd_rf}^6 \geq 0 \quad (\mathbf{Q}_{yhrf}^D) \quad (\text{D.10b})$$

$$\delta_{yhnf}^6 - \mathbf{df}_y \pi_{yhnf}^W \geq 0 \quad (\mathbf{Q}_{yhnf}^{D_s}) \quad (\text{D.10c})$$

D.3.3 Market Clearing between distribution and Storage

The quantity storage manager bought in (demand) equals to the quantity sold from the distributor (supply):

$$\mathbf{Q}_{yhnf}^{W_b} = \mathbf{Q}_{yhnf}^{D_s} \quad (\pi_{yhnf}^W) \quad (\text{D.11a})$$

D.4 Storage

D.4.1 Problem

$$\text{Maximize} \quad : \quad \sum_{\substack{y \in Y \\ h \in H \\ f \in F}} \left\{ \pi_{yhnf}^U \mathbf{Q}_{yhnf}^{W_s} - \pi_{yhnf}^W \mathbf{Q}_{yhnf}^{W_b} - \left(\frac{1}{2} \mathcal{C}_{yhnf}^{W_q} \mathbf{Q}_{yhnf}^W + \mathcal{C}_{yhnf}^{W_l} \right) \mathbf{Q}_{yhnf}^W \right\} \quad (\text{D.12})$$

where U in superscript stands for ultimate or utility price selling to the customers; W_s in superscript stands for the quantity sell out from the warehouse and W_b stands for the quantity buy into the warehouse; W_q and W_l stand for the quadratic and linear storage cost factors; \mathbf{Q}_{yhnf}^W is the quantity of food f stored in year y season h at node n .

The stakeholder is subject to the constraints

$$\mathbf{Q}_{yhnf}^{W_b}, \mathbf{Q}_{yhnf}^{W_s}, \mathbf{Q}_{yhnf}^W \geq 0$$

Storage capacity constraint:

$$\mathbf{Q}_{yhnf}^W \leq \mathbf{Q}_{yhnf}^{W,CAP} \quad (\delta_{yhnf}^8) \quad (D.13a)$$

Balance constraint of storage over years:

For the first season h , if h' is the last season:

$$\mathbf{Q}_{yhnf}^W = \mathbf{Q}_{(y-1)h'nf}^W + \mathbf{Q}_{yhnf}^{W_b} - \mathbf{Q}_{yhnf}^{W_s} \quad (\delta_{yhnf}^{11}) \quad (D.13b)$$

For other seasons:

$$\mathbf{Q}_{yhnf}^W = \mathbf{Q}_{y(h-1)nf}^W + \mathbf{Q}_{yhnf}^{W_b} - \mathbf{Q}_{yhnf}^{W_s} \quad (\delta_{yhnf}^{11})$$

D.4.2 KKT Conditions

$$\pi_{yhnf}^W - \delta_{yhnf}^{11} \geq 0 \quad (\mathbf{Q}_{yhnf}^{W_b}) \quad (D.14a)$$

$$\delta_{yhnf}^{11} - \pi_{yhnf}^U \geq 0 \quad (\mathbf{Q}_{yhnf}^{W_s}) \quad (D.14b)$$

For last season h , where h' is the first season:

$$\mathcal{C}_{yhnf}^{Wq} \mathbf{Q}_{yhnf}^W + \mathcal{C}_{yhnf}^{Wl} + \delta_{yhnf}^8 + \delta_{yhnf}^{11} - \delta_{(y+1)h'nf}^{11} \geq 0 \quad (\mathbf{Q}_{yhnf}^W) \quad (D.14c)$$

For other seasons:

$$\mathcal{C}_{yhnf}^{Wq} \mathbf{Q}_{yhnf}^W + \mathcal{C}_{yhnf}^{Wl} + \delta_{yhnf}^8 + \delta_{yhnf}^{11} - \delta_{y(h+1)nf}^{11} \geq 0 \quad (\mathbf{Q}_{yhnf}^W)$$

D.4.3 Administrative zones to adaptation zones

These are again market clearing equations just as in Sec D.2.3. Here we are converting from administrative zone to adaptation zone instead. With the same definitions,

$$\mathbf{Q}_{yhnf}^{W_s} = \sum_z \mathbf{Q}_{yhnzf}^U \quad (\pi_{yhnf}^U) \quad (\text{D.15a})$$

D.5 Consumers

Consumers are aggregated stakeholders who maximize their utility by consuming food products while minimizing the cost incurred in purchasing the food.

D.5.1 Problem

$$\text{Maximize} \quad : \quad \sum_{\substack{y \in Y \\ h \in H \\ f \in F}} \left\{ \sum_n (\pi_{yhnf}^U \mathbf{Q}_{yhnzf}^U) - \alpha_{yhzf} \mathbf{Q}_{yhzf}^U + \frac{1}{2} \beta_{yhzf} \mathbf{Q}_{yhzf}^U{}^2 \right\} \quad (\text{D.16})$$

The stake holder is subject to the constraints

$$\mathbf{Q}_{yhzf}^U, \mathbf{Q}_{yhnzf}^U \geq 0$$

There could be an export ban or limit where this constraint can become active.

$$\mathbf{Q}_{yhzf}^U \leq \overline{\mathbf{Q}_{yhzf}^U} \quad (\delta_{yhnf}^{20}) \quad (\text{D.17a})$$

Total amount consumed is at most what is purchased from each node

$$\mathbf{Q}_{yhzf}^U \leq \sum_n \mathbf{Q}_{yhnzf}^U \quad (\delta_{yhnzf}^{19}) \quad (\text{D.17b})$$

Consumers have to purchase at least Ψ fraction of their consumption from every corresponding node

$$\mathbf{Q}_{yhnzf}^U \geq \Psi_{nz} \mathbf{Q}_{yhzf}^U \quad (\delta_{yhnzf}^{17}) \quad (\text{D.17c})$$

D.5.2 KKT Conditions

$$\pi_{yhnf}^U - \delta_{yhzf}^{19} - \delta_{yhnzf}^{17} \geq 0 \quad (\mathbf{Q}_{yhnzf}^U) \quad (\text{D.18a})$$

$$\beta_{yhzf} \mathbf{Q}_{yhzf}^U - \alpha_{yhzf} + \delta_{yhzf}^{19} + \delta_{yhnf}^{20} + \sum_n \Psi_{nz} \delta_{yhnzf}^{17} \geq 0 \quad (\mathbf{Q}_{yhzf}^U) \quad (\text{D.18b})$$

Appendix E

Publications and co-authors

- *Sankaranarayanan, S., Feijoo, F., and Siddiqui, S.* (2018). Sensitivity and covariance in stochastic complementarity problems with an application to North American natural gas markets. *European Journal of Operational Research*, 268(1), 25–36.
- *Feijoo, F., Iyer, G., Avraam, C., Sankaranarayanan, S., Wise, M., Clarke, L., and Siddiqui, S.* (2018) Natural gas Infrastructure development - An analysis from GCAM-NANGAM model coupling. *Applied Energy*, 228, 149-166.
- *Basu, A., and Sankaranarayanan, S.* (2018). Can cut generating functions be good and efficient? *Under revision*. arXiv Preprint arXiv:1803.00667.
- *Basu, A., Ryan, C.T., and Sankaranarayanan, S.* (2018) Mixed-integer bilevel representability. *Submitted*. arXiv Preprint arXiv:1808.03865.
- *Sankaranarayanan, S., Zhang, Y., Carney, J., Zaitchik, B., and Siddiqui, S.* Domestic distributional impacts of teff-ban policy and alternatives in Ethiopia. *Working paper*.

- *Basu, A., Molinaro, M., Nguyen, T., and Sankaranarayanan, S.* A machine learning based approach to classify integer programming problems for cut selection. *Working paper.*
- *Basu, A., Ryan, C.T., and Sankaranarayanan, S.* Mixed-integer bilevel solvability. *Working paper.*

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Biography

Sriram Sankaranarayanan was born on 22nd May, 1992 in Coimbatore, India. He completed all his schooling in the nearby town, Mettupalayam.

Following that, Sriram did his undergraduate studies at the Indian Institute of Technology, Kharagpur, where he majored in Civil engineering. Following his graduation in 2013, he spent two years working as an analyst in Deutsche Bank.

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